MEDIUM EFFECTS ON THE BINDING ENERGY OF A HOT HELION (^3He) NUCLEUS

تأثير الوسط على طاقة الربط في نواة الهيليون (^3He) الساخنة

By

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October 1, 2013

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ABSTRACT

The aim of this work is to study the binding energy of the ^3He nuclei moving in a hot low-density vapor of symmetric nuclear matter of protons and neutrons. We are mainly interested in studying the effect of the inclusion of the CM momentum of the ^3He nuclei on their binding energy. The surrounding nucleons are in thermal and chemical equilibrium with the nuclei. We will try to find the Mott density, which is the density of the surrounding vapor at which the binding energy of the ^3He nuclei become zero and so they will dissolve into the surroundings due to the Pauli blocking effect, and how the Mott density will be affected by considering the CM momentum of ^3He nuclei. We found that the existence of protons and neutrons in the vapor surrounding the ^3He nuclei will decrease their binding energy and so they will dissolve into their components and become part of the surrounding vapor. We also found that this dissociation process depends on the temperature. The main conclusion of our work is that the assumption that the ^3He clusters are moving (inclusion of CM momentum) within the surrounding vapor will make them survive to higher densities at the same temperature.

ملخص

ان المادة النووية المحدودة والموجودة في وسط ساخن و قليل الكثافة من البروتونات و النيوترونات تتأثر بهذا الوسط. الهدف من هذا البحث هو دراسة مدى تأثير هذه البروتونات و النيوترونات على استقرار انواية ^{3}He التي تتحرك في هذا الوسط اذا افترضنا أنها موجودة في حالة اتزان حراري وكيميائي مع البخار المحيط المحتوي على هذه البروتونات و النيوترونات. نريد ان نتوصل لكثافة الوسط الذي تتفكك عنده انواية ^{3}He و كيف تختلف هذه الكثافة مع اختلاف درجة الحرارة. توصلنا في هذه الدراسة الى أن وجود البروتونات و النيوترونات في البخار المحيط يعمل على زيادة عدم استقرار انواية ^{3}He فتتفكك وتتلاشى عندما تصل كثافة البخار لحد معين لتصبح جزءا من الوسط المحيط و ذلك يختلف من درجة حرارة لأخرى. و النتيجة الاساسية لهذا البحث هي انه عند الافتراض ان نواة ^{3}He متحركة و ليست ساكنة تستطيع ان تتفادى الاضمحلال حتى كثافات اعلى عند نفس درجة الحرارة.

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CHAPTER 1. INTRODUCTION

One of the most important topics in nuclear physics is to study the composition, stability, and the dynamics of finite nuclear matter (nuclei) under different conditions. Heavy-ion collisions are a very powerful tool to obtain reliable information about nuclear matter behavior over a wide range of densities and temperatures.

Most recent studies in medium energy heavy-ion collisions indicate that there is a possibility that the hot (excited) nuclei resulting from the collision will fragment into small clusters of nucleons. This fragmentation process in the nuclear matter is called the liquid-gas phase transition in which the nuclear matter will transfer from the liquid-like phase to the gaseous one [1, 2].

Neglecting the surface and Coulomb effects (dealing with infinite nuclear matter), it was shown that above or at a special temperature (critical temperature) only the gaseous phase can exist. Below the critical temperature the liquid and gas phases coexist. Attempts to study the liquid-gas phase transition in finite nuclear matter were proposed by Jaqaman *et. al.* in their studies [1, 2]. They concluded that considering the surface effects and the Coulomb force will reduce the critical temperature by a few MeV. They considered, for example, in their work [2] a system of infinite nuclear matter (neglecting both the size effects and the Coulomb force) and found that the critical temperature is about 22.9 MeV. In the same work they also found that for a system composed of 50 protons and 50 neutrons when the Coulomb force is neglected the critical temperature of this system is about 18 MeV. On the other hand, considering the contribution from the Coulomb effect reduces the critical temperature to about 16 MeV.

In 1985 Levit and Bonche [3] studied the effect of the Coulomb force on the liquid-gas phase transition in finite nuclear matter. They showed that above a certain temperature (limiting temperature), which is much lower than the critical temperature of infinite nuclear matter, the charged nucleus is unstable and will fragment into small clusters due to the repulsion between its protons.

In 1989 Jaqaman [4] improved the equation of state considered by Levit and Bonche by considering asymmetric nuclear matter. In another study [5] Jaqaman generalized the equation of state used in [4] by introducing the density-dependent nucleonic effective mass to investigate how it affects the stability of the hot nuclei. He also showed that considering the electric charge of the vapor will raise the value of the limiting temperature.

Most recent investigations [6-14] studied the structure and dynamics of nuclear matter in the vapor phase at very low densities. It was shown that bound states appear in very low density nuclear matter to minimize the energy. These bound states are light clusters such as deuterons, tritons, helions and alpha particles.

In 1988 Jaqaman proposed a study of the nuclear matter at very low temperatures and very low densities [6]. After solving the Hartree-Fock equations of nuclear matter with the Overhauser's orbitals analytically, he noticed that alpha particles are dominant in the low-temperature region (T ≤ 1.1 MeV).

Horowitz and Schwenk in their research [7] in 2006 studied the formation of clusters in low-density nuclear matter. They used the virial expansion to derive the equation of state of low-density nuclear matter composed of protons, neutrons and alpha particles. They studied the composition, entropy, energy, and the symmetry energy. In a recent work [14] the contribution from these clusters was considered while studying the stability of hot charged nuclei. It was concluded that the presence of clusters in the vapor has a significant effect on the stability and so on the limiting temperature.

A nucleus can be considered as a strongly interacting system of fermions. In the normal case nuclear matter is composed of protons and neutrons which are confined within the nucleus of any atom. In this case nuclear matter is in its lowest energy (stable) configuration. The density of nuclear matter in normal form is called the saturation density. In general; the normal saturation density ρ_0 of nuclear matter is estimated to be in the range 0.15-0.17 nucleons/fm^3 [2-6, 12, 13]. At the saturation density the binding energy of nuclear matter is in its maximum value, or nuclear matter is in its minimum-energy state.

The surrounding environment will affect the stability and so the structure and the binding energy of the cluster. At a certain density of nucleons in the surrounding vapor clusters will dissolve and become part of the surrounding environment. This dissolution process is called the Mott effect. The density at which the cluster binding energy vanishes and the cluster dissolves is called the Mott density. Its value depends on the cluster type. For example, it was found in [8] that at T = 0 MeV and assuming that the total momentum of the cluster P = 0, the Mott density of the deuteron is about 10^{-3} nucleons/fm^3 and the Mott density of the alpha particle is about 10^{-2} nucleons/fm^3. The Mott effect occurs as a result of the Pauli blocking effect which is a consequence of the Pauli exclusion principle according to which identical nucleons are prohibited from occupying the same quantum state. This in turn results from the antisymmetrization of the total wave function involving the nucleons inside and outside the nucleus [8-14].

An attempt to study the medium effects on the bound clusters was suggested by Röpke *et al.* [8]. They studied the effect of the surroundings on the stability of the light-clustered nuclear matter from a quantum statistical point of view. They determined the Mott transition density beyond which the many-body clusters dissolve was determined.

A year later Röpke and his coworkers [9] considered the correlation effects in the surrounding medium. Another new step in their work was that they studied the properties of clusters embedded in a hot medium consisting of nucleons and clusters. They assumed that the nucleon-nucleon interaction was of the simple Skyrme type [15].

Beyer *et. al*. [10] considered symmetric nuclear matter which contains equal number of protons and neutrons at a finite temperature to study the properties and distribution of the light clusters up to A = 4. They studied the distribution of the nuclear matter at low-density limit and show that the binding energy of the clusters depends on the density, temperature and the center-of-mass CM momentum of the cluster.

Medium effects in low-density nuclear matter was also studied in [11] by Röpke using the quantum statistical approach. He considered the medium effects on the clusters by describing the self-energy and the Pauli blocking. He found that the Mott density of the alpha particle is about 0.006 nucleons/fm^3 at T = 0 MeV.

A recent study [12] of the medium effects on deuterons, tritons, helions, and helium nuclei was carried out by Typel *et al.*. They calculated the Mott density at different temperatures and for various light nuclei/clusters. All these calculations assumed that these clusters are at rest. They concluded that as the density increases these clusters dissolve when the binding energy decreases to zero mainly because of Pauli blocking effect. They also found that the Mott density increases with temperature regardless of the cluster type. This result is expected as the Pauli blocking is less effective with increasing temperature.

Another recent work by Röpke [13] investigated the stability of light clusters up to A = 4 in hot and dense nuclear matter. Röpke used the quasiparticle approximation to study the dissolution of light clusters due to the Pauli blocking shift. He showed that for the deuteron (A = 2) at zero CM momentum the Mott density increases with temperature. He also noticed that the Pauli blocking effect is less effective when the CM momentum increases at fixed temperature.

We aim in this work to extend the work which was done by Typel and his coworkers [12] by correctly including the CM momentum. In their work they assumed that the ^{3}He nucleus is at rest. The new thing in our work is that we will assume that the ^{3}He nucleus is moving in a hot low-density medium of protons and neutrons which is more realistic.

We will try to find the Mott density at which the binding energy of the ^{3}He nucleus becomes zero and so it will dissolve into the surroundings due to the Pauli blocking. We will use the harmonic oscillator shell model wave function to describe the internal wave function of the nucleus.

After this introduction we will discuss the properties of nucleons and the Skyrme interaction in the next chapter. The nuclear shell model will be the subject of chapter (3). The medium–dependence of the binding energy, which is the main idea of this research, will be discussed in chapter (4). In chapter (5) we used the Fermi Dirac statistics to find the expectation value of the binding energy of ^{3}He nuclei at different temperatures. In chapter (6) we will discuss the results of our research.

CHAPTER 2. PROPERTIES OF NUCLEONS AND NUCLEAR INTERACTION

The nucleus can be described as a medium of strongly interacting nucleons. There are two types of nucleons: protons and neutrons. Each type of nucleus is characterized by its number of protons and neutrons which distinguishes it from others. The study of nucleons and the interaction between them is one of the most active fields in physics.

2.1 PROPERTIES OF NUCLEONS:

At first it was thought that nucleons are elementary particles. Now it is known that each nucleon is made up of three quarks bound together by the so-called strong interaction which is mediated by gluons.

Nucleons (protons and neutrons) are the most common members of the baryon family. The proton is composed of two up quarks and one down quark. On the other hand the neutron consists of one up quark and two down quarks. The difference between their masses is relatively small. It was found that the mass of the proton is 938.272 MeV/c^2 while the mass of the neutron is 939.566 MeV/c^2. [17].

The proton has a charge of $+1e$, where $e = 1.60217733(49)\times10^{-19}$ C is the magnitude of the electric charge of the electron, while the neutron is neutral (has no charge). This can be interpreted depending on the idea that protons and neutrons are composed of quarks which are charged elementary particles. The charge of the up quark is $+2/3e$ and the charge of the down quark is $-1/3e$. This makes it easy now to conclude that the total electric charge of the proton is $+1e$ and of the neutron is 0 [17].

Both protons and neutrons have spin $1/2$ and so they are fermions. This indicates that when nucleons interact with each other the Pauli exclusion principle must be applied and the total wave function must be antisymmetric. We will also see that the spin of nucleons plays an important role in their interactions since the nuclear force is found to be spin-dependent [17].

It is clear from the properties of the proton and the neutron mentioned above that protons and neutrons are similar in most of their properties, for example, both are fermions and the difference between their masses is very small (about 1%). It was also observed that the nuclear force affects protons and neutrons in the same way, thus the proton-proton interaction, neutron-neutron interaction, and neutron-proton interaction are similar. This is the so-called charge-independence property of the nuclear force [17, 19]. The only difference between them is in the electromagnetic properties, thus, if one neglects this difference the proton and the neutron can be treated as two states of the same particle. If we want to adopt this idea another label must be introduced to distinguish between them. This label (operator) is the *isospin* ($\vec{\tau}$) which is mathematically similar to the intrinsic spin ($\vec{s}$) label. Thus, it is easy now to say that the proton and the neutron are two states of the same particle. Using the same mathematics of the intrinsic spin operator we can say that the value of the isospin is $\tau = 1/2$ for the nucleon with two possible values of its third component ($\vec{\tau}$) to distinguish between the proton and the neutron. While one has $\tau_3 = +1/2$ the other has $\tau_3 = -1/2$ [17].

2.2 NUCLEON-NUCLEON INTERACTION:

The nucleon-nucleon interaction is one of the most important issues in nuclear physics. Until now this interaction is not known exactly, so that many attempts were made to suggest reasonable approximations to an effective nucleon-nucleon interaction. As mentioned above nucleons are not elementary particles, thus it is expected that the quarks that make them up will contribute in the interaction between nucleons. Since quarks cannot be found in isolation the interaction between nucleons results from the exchange of mesons which are quark-antiquark pairs.

As a first step in understanding the nature of the nucleon-nucleon interaction it is convenient to make use of the simplicity of the deuteron as it contains only two nucleons (a proton and a neutron) loosely bound to each other.

2.2.1 THE DEUTERON:

The deuteron is a very simple bound system with special properties. The binding energy of the deuteron is $2.22457312\ MeV$ and its mass is $1876.1244\ \ MeV/c^2$. It is also known that the parity of the deuteron structure is positive [17]. Here are the measured values of some of the most common observed properties of the deuteron listed in Table 1

Table 1. The observed properties of the deuteron in its ground state[1]

Ground-state property	**Value**
Binding energy	2.22457312(22) MeV
Spin and parity, J^{π}	1^{+}
Magnetic dipole moment, μ_d	0.857438230(24) μ_N
Electric quadrupole moment, Q_d	0.28590(30) e.fm^2

As mentioned above the deuteron consists only of one proton and one neutron whose intrinsic spins can add up to 1 or 0. This means that we have two possibilities of the intrinsic spin (S) of the deuteron: S = 0 (singlet state) and S = 1 (triplet state). But it was found that the deuteron exists only in the triplet state. This can be shown by making use of the fact that the deuteron parity is positive. To be more clear let us separate the wave function of the deuteron into three parts: the intrinsic wave function of the proton, the intrinsic wave function of the neutron, and the third part is the orbital wave function which represents the relative motion of the proton and the neutron. This separation process makes it easy for us to detect that the parity of the

[1] Note that the numbers given in parentheses represent the uncertainties in the last digits of the measured values. Also $\mu_N = \frac{e\hbar}{2m_p}$ is the nuclear magneton, where $\hbar$ is the reduced Planck constant and m_p is the mass of the proton.

deuteron is only associated with the parity of the orbital wave function of the relative motion as the intrinsic parities of the wave functions of both the proton and the neutron are identical (two states of the nucleon). But the orbital wave function is given by the spherical harmonics $Y_L^M(\theta,\varphi)$ whose parities are given by the factor $(-1)^L$, and so we can now use the fact that the parity of the deuteron is positive to conclude that the orbital angular momentum L must be even [17].

Depending on the above discussion and also making use of the fact that the total spin of the deuteron in its ground state is J = 1, where J = S + L, it is easy to conclude that the allowed values of the orbital angular momentum L are 0 or 2. It is clear now that it is impossible for the deuteron to be in the singlet S = 0 state, as when S = 0 and L is even we will never get J = 1, and as a result the spin angular momentum of the deuteron is S = 1 (triplet state).

The deuteron in its ground state still has two possibilities, L = 0 and L = 2, for the space part of the wave function. This means that both the $^3S_1$ (triplet S state) and $^3D_1$ (triplet D state) components appear in the deuteron wave function. These facts about the deuteron give a good indication that the nuclear force mixes different L-components [17, 18].

The existence of both $^3S_1$- and $^3D_1$- states in the deuteron ground state can also be verified depending on other observed properties of the deuteron, such as the electric quadrupole and the magnetic dipole moments. It is known that the electric quadrupole operator measures the departure from a spherical charge distribution of a nucleus. Since the state ${}^3S_1$ is spherically symmetric, the observed positive value of the deuteron electric quadrupole moment ($Q_d = 0.28590\, e.fm^2$) is a strong evidence for the presence of the ${}^3D_1$-component in the deuteron ground state [17].

The magnetic dipole moment of the deuteron μ_d also indicates that there is a small admixture of the ${}^3D_1$-component in the deuteron ground state. At first let us assume that there is only the ${}^3S_1$ ($L = 0$) state in the deuteron ground state. In this case the deuteron magnetic dipole moment results only from the sum of the intrinsic dipole moments of a proton μ_p and a neutron μ_n [17], thus

$$\mu_d({}^3S_1) = \mu_p + \mu_n = 0.87976\, \mu_N \tag{2.1}$$

The observed value of μ_d is $0.857483\mu_N$ (see Table 1) which is slightly different from the value obtained from eq. (2.1). This leads us to conclude that the ground state of the deuteron is not a pure state and it is an admixture of ${}^3S_1$ and ${}^3D_1$ states. Now the deuteron wave function can be written as

$$|\psi_d\rangle = a|{}^3S_1\rangle + b|{}^3D_1\rangle \tag{2.2}$$

with the normalization condition

$$a^2 + b^2 = 1 \tag{2.3}$$

Eq. (2.3) together with the observed experimental values of the magnetic dipole moment μ_d and the electric quadrupole moment Q_d can be used to obtain the values of a and b. The value of a^2, which represents the probability that the deuteron is in the $^3S_1$ state, is about 96%. On the other hand the probability of being in the $^3D_1$ state (b^2) is 4% [19]. This means that the deuteron ground state wave function is a linear combination of the $^3S_1$ state (the dominant state) and the $^3D_1$ state. The very small admixture of the $^3D_1$-component in the deuteron ground state plays an important role in the study of the properties of the nuclear force as we shall see later.

2.2.2 PROPERTIES OF THE NUCLEON-NUCLEON INTERACTION:

The above discussion of the deuteron properties indicates that the deuteron is a rich source for gathering important information about the properties of the nucleon-nucleon interaction which in turn plays a vital role in understanding the nuclear force.

The dominant existence of the triplet spherically symmetric state in the deuteron wave function indicates that the nucleon-nucleon potential contains a dominant central spherically symmetric term $V_c(r)$ which depends only on r. This central attractive term is responsible for holding nucleons together within the nucleus as it can overcome the central Coulomb repulsion between protons [17, 19].

As mentioned above, the small admixture of $^3D_1$ state in the dominant spherically symmetric $^3S_1$ state in the deuteron ground state indicates that the nucleon-nucleon interaction is not purely central as it contains different values of L-components. This means that a small non-central tensor component must be added to the central dominant force between two nucleons. The tensor force term depends mainly on the separation position vector $\vec{r}$ and the spins of the two nucleons $\vec{s}_1$ and $\vec{s}_2$. This is not strange as a nucleon is mainly characterized by its spin. Thus the tensor force depends on the scalar product of $\vec{r}$ and the spin $(\vec{s}.\vec{r})$, or the cross product of them $(\vec{s} \times \vec{r})$ as these are the only terms relating $\vec{r}$ and $\vec{s}$ with each other. Thus the general form of the tensor character of the nucleon-nucleon interaction can be written as $S_{12} = \frac{3(\vec{s}_1.\vec{r})(\vec{s}_2.\vec{r})}{r^2} - \vec{s}_1.\vec{s}_2$. If we take the average over all angles the tensor force term S_{12} will vanish. This means that this term is negligible if we consider finite nuclear matter with many nucleons [17, 18, 19].

Other restrictions on the nucleon-nucleon interaction can also be concluded by considering the symmetry requirements on a two-nucleon system. For example, it was assumed that the nucleon-nucleon interaction is charge symmetric. This means that after subtracting the Coulomb interaction between a pair of protons, the proton-proton and the neutron-neutron interactions are similar [19]. Also the nucleon-nucleon interaction is assumed to be charge-independent, that is the proton-proton, neutron-neutron, and the proton-neutron interactions are assumed to be equal. This assumption is usually made although there is a small difference $(\approx 1\%)$ between the proton-neutron interaction and the interaction between a pair of protons or a pair of neutrons [17].

The invariance of nucleon-nucleon interaction under the translation of the whole system of the two nucleons in space is another symmetry restriction in the nuclear interaction. This means that the interaction between two nucleons depend only on the relative position of the two nucleons with respect to each other $\vec{r} = \vec{r}_1 - \vec{r}_2$ and not on their absolute positions ($\vec{r}_1$ and $\vec{r}_2$). The nucleon-nucleon interaction can also depend on the relative momenta of the two nucleons $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$. This property is known as the Galilean invariance of the two nucleon system. Beside the translational and Galilean invariances, the nucleon-nucleon interaction also satisfies other symmetries, such as, the invariance under a rotation of the coordinate system or a permutation between the two nucleons, time reversal $(t \rightarrow -t)$, and parity invariance [17].

The dependence of the two-nucleon interaction on the relative momenta of the two nucleons will add a new term to the nucleon-nucleon interactions. This term is called the spin-orbit term as it depends on both the total intrinsic spin $\vec{S} = \vec{s}_1 + \vec{s}_2$ and the total relative orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ of the two-nucleon system. The general form of the spin-orbit term, which satisfies both the parity and time reversal invariances, is $V_{so}(r)\vec{L}.\vec{S}$, where $V_{so}(r)$ is some function of r [17, 19].

Another term, which is a spin-dependent one, must be added to the nucleon-nucleon potential. The existence of such term is mainly predicted depending on the fact that only the triplet (S = 1) state exists in the deuteron. This term must also satisfy the parity and translation invariances. The general form of this term is

$V(r)(1+x_0P_\sigma)$, where $V(r)$ is an arbitrary function of r and x_0 is any constant which is determined by fitting the experimental data. Also in this term $P_\sigma = \frac{1}{2}\left(1+\frac{4}{\hbar^2}\vec{s}_1.\vec{s}_2\right)$ is the spin exchange operator which acts on the spin states of the two-nucleon system. When P_σ acts on the triplet state it gives +1 as this state is symmetric under the exchange of the spins of the two nucleons. On the other hand, when it acts on the singlet spin state, which is antisymmetric, it gives -1 [18].

Although the study of the two-nucleon systems, such as the deuteron, is an important source of gathering information about the nuclear force, it is on the other hand not enough or in other words another source must be used in order to obtain better results. Thus many experimental approaches, in which one nucleon is scattered off another one, were made to achieve this purpose. The results of nucleon-nucleon scattering introduced a new progress in the understanding of the nuclear force.

By analyzing the nucleon-nucleon scattering results at different ranges of energies it was noticed that the nucleon-nucleon interaction can be divided into three parts. The short-range repulsive part ($r \leq 1$ fm) indicates that the nuclear force must contain a repulsive hard core. The intermediate-range part (1 fm $< r < 2$ fm) and the long-range part ($r > 2$ fm) [17] are generally attractive. The presence of the short-range repulsive part in the nucleon-nucleon interaction is consistent with the known fact that the nuclear density is roughly constant which means that there is something preventing nucleons from being very close to each other.

Depending on the results of the nucleon-nucleon scattering experiments it was also possible to find evidence for other properties of the nucleon-nucleon interaction. For example, it was found that the proton-proton scattering parameters (after subtracting the effect of the Coulomb force) are approximately similar to those of neutron-neutron scattering, and thus, it is convenient to describe the nucleon-nucleon interaction to be charge symmetric [19].

In summary, one can say that studies of the deuteron properties, the symmetries of the two-nucleon systems, and the nucleon-nucleon scattering experiments enable us to suggest a reasonable potential describing the nucleon-nucleon interaction. The dominant term in this potential is a central attractive potential with a repulsive hardcore. There are also small contributions to the nucleon-nucleon potential that include a non-central tensor force component, a spin-orbit term, and a spin-dependent term.

2.2.3 NUCLEON-NUCLEON POTENTIALS:

In the 1930's Yukawa proposed the first step towards a fundamental theory of the nucleon-nucleon interaction. He followed a physical mechanism to formalize his theory. Yukawa made an analogy with the quantum electrodynamics which says that the electromagnetic interaction between charged particles is contributed by the exchange of virtual photons, and thus, he predicted that the nucleon-nucleon interaction is mediated by the exchange of virtual particles (mesons). By solving the time-independent Klein-Gordon equation,

$$\nabla^2 \phi(r) = \frac{m^2 c^2}{\hbar^2} \phi(r) - g\delta(r) \tag{2.4}$$

Yukawa obtained a general form of the nucleon-nucleon potential which is called the Yukawa potential

$$\phi(r) = \frac{g}{4\pi} \frac{e^{-mcr/\hbar}}{r} \tag{2.5}$$

where g is an adjustable coupling constant, $\hbar$ is the reduced Planck constant, c is the speed of light, and m is the mass of the exchanged meson. The propagation process of mesons between nucleons must satisfy the uncertainty principle. This means that if only one meson is exchanged then its means that the energy of the transferred meson and its life time is related through the equation [20]:

$$(mc^2)\Delta t \approx \hbar \tag{2.6}$$

From this equation we can find that the range of the force, which results from the exchange of a meson of mass m between two nucleons, is given by

$$range = c\Delta t \approx \frac{\hbar}{mc} \tag{2.7}$$

It is easy now, from eq. (2.7) , to notice that the main factor that affects the range of the nucleon-nucleon interaction is the mass of the exchanged meson. As we can see from this relation the heavier the mass, the shorter is the range. We can see also that eq. (2.5) can be written in terms of eq. (2.7) as,

$$\phi(r) = \frac{g}{4\pi}\frac{e^{-r/range}}{r} \tag{2.8}$$

which indicates that the Yukawa potential shows that the range of the nuclear potential depends strongly on the mass of the exchanged meson.

The discovery of the pion (π meson) in 1947 [26] supported the idea proposed by Yukawa. If we substitute the mass of the pion ($\approx 140\ MeV/c^2$) in eq. (2.7), we will obtain the observed value of the nuclear force range which is about 1.4 fm. As mentioned above the nucleon-nucleon interaction can be divided mainly into short-, intermediate-, and long-range parts. By fitting the experimental data it was noticed that Yukawa's idea of simple one-pion exchange potential (OPEP) gives reasonable results for the long-range part [17].

In attempting to construct a potential that has the correct form for long-, intermediate-, and short-range parts a phenomenological approach was adopted. This approach was mainly based on generalizing the one-pion exchange (OPE) idea proposed by Yukawa to the so-called one-boson exchange (OBE) idea [17]. According to this idea the long-range part is contributed by pions while the intermediate-range part is made of single heavier mesons. The short-range repulsive part comes from the exchange of heavy mesons, such as the ρ-, ω-,and η-mesons [18].

The masses and the range of exchange of some of the mesons are summarized in Table 2

Table 2. Masses and exchange ranges of some mesons that contribute to the nucleon-nucleon interaction.

Meson Particle	**Particle Mass ($\mathbf{MeV/c^2}$)**	**Range (fm)**
π-meson	140	1.4
η-meson	549	0.35
ρ-meson	769	0.25
ω-meson	783	0.25

It is also possible to consider the exchange of more than one meson to obtain the intermediate- and short- ranges of the nucleon-nucleon potential. This can be interpreted easily by rewriting eq. (2.7) for the case in which n mesons are exchanged. If we do this we will get

$$range = c\Delta t \approx \frac{\hbar}{nmc} \tag{2.9}$$

Now it is clear, from eq. (2.9), that as the number of exchanged mesons increases the range of the interaction will be shorter. In general, the intermediate-range is dominated by two-pion exchange while the short-range part results from multipion exchanges. This can give us a good indication to the many-body part of the nuclear force. For example, if a nucleon emits two mesons, they may be absorbed by two different nucleons, and this in turn indicates the existence of three-body forces [17, 20]. The ^{3}He nucleus, which we will study in our work, is a three-nucleon system (contains two protons and one neutron) and so two- and three-body interactions must be considered in this case as we will see below.

Although this approach is successful in many applications, it still has some difficulties as we have different mesons with different masses. Also this approach assumes that mass is an adjustable parameter which contradicts with the properties of real mesons [17].

Another approach to formalize the nucleon-nucleon potential is based on effective field theory (EFT). This approach appeared after the discovery of quantum chromodynamics (QCD). It was useful especially in describing the short-range part of the nuclear potential [21, 22].

One of the potentials derived from the EFT is the Paris potential [23] which was introduced by the Paris group in the 1970's. The Paris group interpreted the nucleon-nucleon interaction in the long- and intermediate ranges by the one- and two-pion exchange respectively. For the hard core they said that it mainly results from the exchange of three and four pions. Another group, whose work is approximately similar to the work of the Paris group, is the Bonn group [24]. The main difference between the two groups is in the implementation of the short-range nucleon-nucleon interaction. In the Bonn potential the repulsive short-range interaction is mainly contributed by the ω-meson exchange [25].

Recently, approximately in the 1990's, new phenomenological potentials were introduced to represent the nucleon-nucleon interaction. These potentials are called high-precision charge-dependent nucleon-nucleon potentials. The Argonne V18, the Nijmegen potentials (Nijm I, Nijm II), and the Idaho potential are examples of such potentials [22, 25, 26, 27].

2.3 EFFECTIVE NUCLEON-NUCLEON INTERACTION INSIDE NUCLEAR MATTER:

The interaction between a pair of nucleons when they are free is different from that when they are near other nucleons as when they are bound within a nucleus. Thus, it is insufficient to specify the nucleon-nucleon potential completely by studies made on systems of two free nucleons alone. Nucleons within a nucleus interact with each other not only via two-body interactions, there are also three-, four-, and higher-rank interactions. The ^{3}He nucleus, for example, which is the subject of our research is a three-nucleon system and so a three-body force appears in it [21].

The effective nucleon-nucleon interaction for bound nucleons is not known exactly. Many attempts were made to suggest reasonable approximations to it. The Brueckner approach in the 1950s introduced one of the most important effective interactions in nuclear physics. Via this approach, which is mainly based on perturbation theory, Brueckner theory provides a very powerful tool to interpret the existence of the repulsive hard core by describing the interaction between a pair of nucleons when they are surrounded by other nucleons [21, 27, 28]. A study by Eden [29] used the Bruckner theory to study the structure and properties of finite nuclear matter. The success of the Bruckner-based approach in some aspects does not mean that it is simple. On the contrary, it is so complex and needs a lot of detailed work especially when used to study finite nuclei with a large number of nucleons.

As an alternative, a phenomenological simple effective interaction was first proposed by Skyrme in 1959 [15]. The Skyrme interaction which is a zero-range effective nucleon-nucleon interaction was successfully used by Vautherin and Brink [16] in their study of the properties of nuclear matter and some closed-shell nuclei. The introduction of this zero range nucleon-nucleon interaction simplifies nuclear calculations and makes them easier.

2.4 THE SKYRME INTERACTION:

The original form of this interaction as in [16] is

$$v = \sum_{i<j} v_{ij} + \sum_{i<j<k} v_{ijk} \qquad (2.10)$$

where,

v_{ij} represents the mostly attractive two-body interaction and,

v_{ijk} represents the repulsive three-body interaction.

The general form of the two- and three-body interactions as used by Vautherin and Brink in their study [16] are

$$v_{12} = -t_0(1 + x_0 P_\sigma)\delta(\vec{r}_1 - \vec{r}_2) + \frac{1}{2}t_1\left[\delta(\vec{r}_1 - \vec{r}_2)k^2 + k'^2\delta(\vec{r}_1 - \vec{r}_2)\right] +$$

$$t_2\vec{P}'.\,\delta(\vec{r}_1 - \vec{r}_2)\vec{K} + iW_0(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\ \vec{P}' \times \delta(\vec{r}_1 - \vec{r}_2)\vec{P} \qquad (2.11)$$

$$v_{123} = t_3\delta(\vec{r}_1 - \vec{r}_2)\delta(\vec{r}_2 - \vec{r}_3) \qquad (2.12)$$

where $\vec{r}_1$, $\vec{r}_2$and $\vec{r}_3$ are the position vectors of the nucleons, $\vec{P} = \frac{\vec{\nabla}_1 - \vec{\nabla}_2}{2i}$ and $\vec{P}' = -\frac{\vec{\nabla}_1 - \vec{\nabla}_2}{2i}$ represent the relative momentum operators acting on the right and acting on the left respectively and $W_0 = -\frac{2}{3}\pi\int_0^{+\infty} V_{LS}(r)r^4 dr$, where $V_{LS}(r)$ is the two-body spin-orbit potential at the short-range limit. The Skyrme interaction parameters x_0, t_0, t_1, t_2 and t_3 are determined by fitting the properties of infinite nuclear matter and some finite nuclei.

Vautherin and Brink also showed that for even-even nuclei three-body interactions are equivalent to a two-body density-dependent interaction

$$V_{12} = \frac{1}{6}t_3(1 + P_\sigma)\rho\left[\frac{\vec{r}_1 + \vec{r}_2}{2}\right]\ \delta(\vec{r}_1 - \vec{r}_2) \qquad (2.13)$$

where ρ is the density. But in the case of ^{3}He this result cannot be used because of the unpaired neutron.

For simplicity we will neglect the momentum dependent terms in (2.11). This means that the form of the Skyrme interaction which we will use in this thesis is

$$v_{12} = -t_0(1 + x_0 P_\sigma)\delta(\vec{r}_1 - \vec{r}_2), \text{ and } v_{123} = t_3\delta(\vec{r}_1 - \vec{r}_2)\delta(\vec{r}_2 - \vec{r}_3) \qquad (2.14)$$

These are the two- and three-body interaction terms which we will use in our work.

CHAPTER 3. NUCLEAR SHELL MODEL

Several models have been proposed to study the structure of nuclei, with each method being based on a set of simplifying assumptions. Each model is successful and gives good results in a limited range, but fails when applied to data outside of this range. There are three nuclear structure models that were introduced to describe the internal working of the nucleus. These models are: the *Collective Model*, the *Liquid-Drop Model,* and the *Nuclear Shell Model* [17]. We will be using the Shell Model.

3.1 EVIDENCE AND BASIC ASSUMPTIONS OF THE NUCLEAR SHELL MODEL

The nuclear shell model is one of the most important and useful models of nuclear structure. The original motivation for proposing it was that it was found experimentally that nuclei with certain numbers of protons or neutrons have relatively high nuclear binding energies per nucleon. This means that these nuclei are more stable than others. These numbers were called the "magic numbers" as they were not understood at that time. This situation was similar to the extreme stability of the noble gases which was explained successfully by the existence of completely filled shells in the electronic shell model. This was the main reason which encouraged scientists to adopt the idea of shells to describe nuclear structure[19, 30, 31].

The basic hypotheses of the nuclear shell model are similar to those of the electronic shell model. This means that protons and neutrons within the nucleus are constrained to move in shells with each shell being limited to a certain maximum number of protons or neutrons. It is known that nucleons are confined within the nucleus via the nuclear force which is a short-range force as mentioned in the previous chapter. When the distance between nucleons is very small the nuclear force becomes repulsive. This nucleon-nucleon repulsive force and the Pauli exclusion principle contribute to the independent motion of the nucleons within the very dense nuclear matter. The independent motion of protons and neutrons within the nucleus is a basic assumption of the nuclear shell model. Thus, the nuclear shell model is called the Independent-particle model [30, 32]. This independent motion of nucleons in turn reduces the many-body problem into many single particle problems.

Similar to the electronic shell model, when the outer shell is filled, the nucleus will have extreme stability. This is what happens when the number of protons or neutrons or both is magic. Thus, the shell mode was able to give a reasonable interpretation of the extra stability of some nuclei. On the other hand, the properties of neighbouring nuclei with a closed shell plus or minus one nucleon are determined by the unpaired nucleon. This effectively reduces the many-body problem to a single-particle problem [31].

The idea of the nuclear shell model was first suggested in the 1930's. This model was able to give a reasonable interpretation of the high stability of nuclei containing the first four magic numbers 2, 8, 20, and 40. In 1949, Mayer tried to improved the conventional shell model to account for the higher magic numbers by introducing the strong spin-orbit force for individual nucleons. In the same year, Jensen and his colleagues Axel and Suess studied the shell model independently from Mayer and they reached the same result. Mayer and Jensen won the Nobel Prize because of the importance of the improvement they introduced in the shell model [30, 31, 32]. In 1954 Brueckner improved the idea of independent motion of particles depending on the short range nucleon-nucleon repulsion and the Pauli exclusion principle [30].

3.2 Center-of-Mass Spurious States In The Nuclear Shell Model

The nucleus is a finite self-bound system. Nucleons in the nucleus can be treated as forming a many-body system. This system has both internal motion and center-of-mass motion. While studying the nuclear dynamics of nuclei using the nuclear shell model, the basic problem was the appearance of spurious center-of-mass (CM) motion. As mentioned above the shell model assumes that nucleons move independently in a potential well $V(\vec{r})$. Assuming that this nuclear potential well is centered at the origin, then in the ground state the CM oscillates in this potential with the zero point motion. The shell model wave function of the excited states may lead to the CM oscillating with higher energies. This is called spurious CM motion. This means that to remove the contribution of this spurious motion it is necessary to split the many-body motion into center-of-mass motion and internal motion.

Center-of-mass spurious motion in the shell model calculations was one of the active fields of research since the 1950's. Many ways were suggested to solve the problem [33-42]. An effective way to separate the center-of-mass motion from the internal motion is to carry out the nuclear shell model calculations in the harmonic oscillator representation as first shown by Elliott and Skyrme in their work in 1955 [34].

3.3 HARMONIC OSCILLATOR NUCLEAR SHELL MODEL:

As the nuclear shell model is based on the independent motion of nucleons, the shell model wave function $\Psi_{SM}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_A)$ can be written as the product of the single-particle wave functions $\psi_i(\vec{r}_i)$. But nucleons within the nucleus are fermions which means that the total wave function must be antisymmetric. Thus, to ensure the antisymmetrization property the nuclear shell model wave function is written in the form of a Slater determinant,

$$\Psi_{SM}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_A) = \frac{1}{\sqrt{A!}} det \begin{vmatrix} \psi_1(\vec{r}_1) & \psi_1(\vec{r}_2) & \dots & \psi_1(\vec{r}_A) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_A(\vec{r}_1) & \psi_A(\vec{r}_2) & \dots & \psi_A(\vec{r}_A) \end{vmatrix} \tag{3.1}$$

where A is the number of nucleons, and $\vec{r}_i$ are the coordinates of the individual nucleons. The average interaction between a nucleon and all other nucleons can be expressed as a single-particle potential $V(\vec{r})$ which depends on the position of the nucleon. Different forms of the nuclear potentials were suggested. The harmonic oscillator potential well is one of these nuclear potentials. Assuming that the harmonic oscillator potential well is centered at the origin, $V(\vec{r}_i) = \frac{1}{2}m\omega^2(\vec{r}_i)^2$, the shell model wave function can be written as:

$$\Psi_{SM_n} = |\psi_{n1}(\vec{r}_1)\psi_{n2}(\vec{r}_2) \ldots \psi_{nA}(\vec{r}_A)| \tag{3.2}$$

where $\psi_n(\vec{r}_i) = N_n H_n(\beta\vec{r}_i)e^{-\frac{\beta^2(\vec{r}_i)^2}{2}}$, $N_n = \left(\frac{\beta}{\pi^{1/2}2^n n!}\right)^{1/2}$, H_n is the Hermite Polynomial of order n, and $\beta^2 = m\omega/\hbar$, where ω is the angular frequency of the harmonic oscillator. For the sake of simplicity, antisymmetrization of the wave function is not explicitly shown in eq. (3.2) and the following equations, but it is understood that antisymmetrization will be carried out in the actual calculation. The shell model wave function in eq. (3.2) still contains the unphysical vibration of the center-of-mass within the potential well, as will be shown below. Elliott and Skyrme [34] showed that the harmonic oscillator representation of the shell model wave function made it easy to separate the spurious CM motion. Taking the origin of the harmonic oscillator potential well at $\vec{R}$, where $\vec{R} = \sum_{i=1}^{A} \vec{r}_i/A$, and using the following change of variables

$$\vec{\eta}_1 = \vec{r}_1 - \vec{R},\ \vec{\eta}_2 = \vec{r}_2 - \vec{R}\ ,\ldots,\ \vec{\eta}_A = \vec{r}_A - \vec{R}. \tag{3.3}$$

the wave function (3.2), can be written as

$$\Psi_{SM_n} = N_{n1}N_{n2} \dots N_{nA}H_{n1}\left(\beta\left(\vec{\eta}_1 + \vec{R}\right)\right)H_{n2}\left(\beta\left(\vec{\eta}_2 + \vec{R}\right)\right) \dots H_{nA}\left(\beta\left(\vec{\eta}_A + \vec{R}\right)\right)$$

$$\times\, e^{-\frac{\beta^2}{2}\left[\left(\vec{\eta}_1+\vec{R}\right)^2+\left(\vec{\eta}_2+\vec{R}\right)^2+\cdots+\left(\vec{\eta}_A+\vec{R}\right)^2\right]}$$

$$= N_{n1}N_{n2} \dots N_{nA}H_{n1}\left(\beta\left(\vec{\eta}_1 + \vec{R}\right)\right)H_{n2}\left(\beta\left(\vec{\eta}_2 + \vec{R}\right)\right) \dots H_{nA}\left(\beta\left(\vec{\eta}_A + \vec{R}\right)\right)$$

$$\times\, e^{-\frac{\beta^2}{2}\left[\eta_1^2+\eta_2^2+\cdots+\eta_A^2+AR^2\right]}$$

$$= N_{n1}N_{n2} \dots N_{nA}H_{n1}\left(\beta\left(\vec{\eta}_1 + \vec{R}\right)\right)H_{n2}\left(\beta\left(\vec{\eta}_2 + \vec{R}\right)\right) \dots H_{nA}\left(\beta\left(\vec{\eta}_A + \vec{R}\right)\right)$$

$$\times\, e^{-\frac{\beta^2}{2}\left[\eta_1^2+\eta_2^2+\cdots+\eta_A^2\right]}e^{-\frac{A}{2}\beta^2R^2}$$

It is clear now that the term $e^{-\frac{A}{2}\beta^2R^2}$ represents the CM vibration within the harmonic oscillator potential well. This vibration is not physical and so must be removed. Thus, the usual shell model wave function is a product of the internal wave function which represents the internal motion of nucleons within the oscillator, and the center-of-mass wave function which represents the unphysical oscillatory motion of the CM itself in the oscillator potential. Thus,

$$\Psi_{SM_n} = \Psi_{int_n}\Psi_{CM_n} \tag{3.4}$$

The total momentum and the total angular momentum of any nucleus are conserved because the Hamiltonian of the nuclear system is invariant under translation and rotation [35, 37]. As mentioned above the nuclear wave function can be written as in (3.4). In this formula the internal wave function is invariant under the translation, while the center-of-mass wave function is not. This means that the center-of-mass wave function is not an eigenstate of the nuclear Hamiltonian. Thus, the contribution from the oscillatory center-of-mass motion is not physical and must be removed. After the elimination of this CM oscillatory motion the shell model wave function in its correct form is

$$\Psi_{SM_n} = N_{n1}N_{n2} \dots N_{nA} H_{n1}\left(\beta\left(\vec{r}_1 - \vec{R}\right)\right) H_{n2}\left(\beta\left(\vec{r}_2 - \vec{R}\right)\right) \dots H_{nA}\left(\beta\left(\vec{r}_A - \vec{R}\right)\right)$$

$$\times\, e^{-\frac{\beta^2}{2}\left[\left(\vec{r}_1-\vec{R}\right)^2+\left(\vec{r}_2-\vec{R}\right)^2+\cdots+\left(\vec{r}_A-\vec{R}\right)^2\right]} \qquad (3.5)$$

To be more clear let us write the harmonic oscillator shell model Hamiltonian

$$H_{SM} = \sum_{i=1}^{A} \frac{-\hbar^2}{2m} \nabla_i^2 + \frac{1}{2} m\omega^2 r_i^2 \qquad (3.6)$$

where $\vec{r}_i$ and $\vec{p}_i$ are the coordinates and momenta of the individual nucleons. Using the change of variables as in [37] the harmonic oscillator Hamiltonian can be separated as

$$H_{SM} = \left(\frac{-\hbar^2}{2mA} \nabla_{\xi_0}^2 + \frac{1}{2}(mA\omega^2)\xi_0^2\right) + \sum_{j=1}^{A-1}\left(\frac{-\hbar^2}{2m_j} \nabla_{\xi_j}^2 + \frac{1}{2}\left(m_j\omega^2\right)\xi_j^2\right)$$

where $\vec{\xi}_i = \vec{r}_i - \frac{1}{A-i}\sum_{j=i+1}^{A} \vec{r}_j$ with $\vec{r}_0 = 0$, and $m_j = (A-j)m/(A-j+1)$ is the reduced mass of the j_{th} nucleon.

$$\Rightarrow H_{SM} = H_{CM} + H_{int} \qquad (3.7)$$

with

$$H_{CM} = \frac{-\hbar^2}{2mA} \nabla_{\xi_0}^2 + \frac{1}{2}(mA\omega^2)\xi_0^2 \qquad (3.8)$$

We can see that the CM oscillates in a potential $\frac{1}{2}mA\omega^2\xi_0^2$ with an angular frequency ω', given by $\omega' = \sqrt{\frac{mA\omega^2}{mA}} = \omega$. This means that the minimum CM energy quantum is $\frac{3}{2}\hbar\omega$.

Now, both H_{CM} and H_{int} obey the Schrödinger equation

$$(H_{CM} - E_{CM})\Psi_{CM} = 0 \tag{3.9}$$

and

$$(H_{int} - E_{int})\Psi_{int} = 0 \tag{3.10}$$

Thus, the total energy of the nuclear system is

$$E = E_{CM} + E_{int} \tag{3.11}$$

When all the nucleons are in the lowest energy level of the harmonic oscillator the CM oscillatory motion does not result in spurious states, but gives an unphysical value of $\frac{3}{2}\hbar\omega$ which must be subtracted. On the other hand, for excited states it may contribute even higher spurious amounts [39]. This means that the states are spurious if their oscillatory CM energy E_{CM} is larger than $\frac{3}{2}\hbar\omega$ [37, 42]. A study of the excited states in ^{4}He [40] showed that the center-of-mass motion will affect the calculated binding energy of the nucleus. Even in the ground state we need to eliminate the $\frac{3}{2}\hbar\omega$ from the total energy of the nucleus.

3.4 ^{3}He CALCULATIONS USING HARMONIC OSCILLATOR NUCLEAR SHELL MODEL:

In our work we will study the stability of ^{3}He nucleus (helion) when it is immersed in a vapor of nucleons. Here is a list of some of the properties of an isolated ^{3}He nucleus [44] in Table 3 below

Table 3. Observed properties of an isolated ^{3}He nucleus

Property	**Value**
Binding energy (B_0)	7.718 MeV
Spin, J	$1/2$
Mass	2809.41MeV/c^2
rms radius	1.76 fm

The ^{3}He nucleus consists of two protons and one neutron. Thus, using the Harmonic oscillator shell model representation, the correct wave function of the ^{3}He nucleus in the ground state can be derived as in (3.5) to be

$$\Psi(123) \;=\; Ae^{-\frac{\beta^2}{2}\left[(\vec{r}_1-\vec{R})^2+(\vec{r}_2-\vec{R})^2+(\vec{r}_3-\vec{R})^2\right]}\psi_{spin}(123) \qquad (3.12)$$

where A is the normalization constant. $\vec{r}_1$, $\vec{r}_2$and $\vec{r}_3$ are the position vectors of the two protons and the neutron respectively. Also in eq. (3.12) $\vec{R} = \sum_{i=1}^{3} \vec{r}_i/3$. This wave function does not include any unphysical oscillatory CM motion. Considering the case that the CM of the ^{3}He nucleus is not at rest, but moving with momentum $\hbar\vec{K}$, within a cubic box of edge L the wave function will be

$$\Psi(123) = \frac{A}{L^{3/2}} e^{-\frac{\beta^2}{2}\left[(\vec{r}_1-\vec{R})^2+(\vec{r}_2-\vec{R})^2+(\vec{r}_3-\vec{R})^2\right]} e^{i\vec{K}.\vec{R}} \psi_{spin}(123) \qquad (3.13)$$

where $\vec{K}$ represents the wave vector of the ^{3}He nucleus.

As the ^{3}He problem is a three body problem we will change the variables as in [43]

$$\left(\vec{r}_1-\vec{R}\right)^2+\left(\vec{r}_2-\vec{R}\right)^2+\left(\vec{r}_3-\vec{R}\right)^2 = \frac{6}{9}\left(r_1^2+r_2^2+r_3^2-\vec{r}_1.\vec{r}_2-\vec{r}_1.\vec{r}_3-\vec{r}_2.\vec{r}_3\right)$$

$$= \frac{2}{3}\left(\frac{3}{4}r_1^2+\frac{3}{4}r_2^2-\frac{3}{2}\vec{r}_1.\vec{r}_2+\left(\vec{r}_3-\frac{\vec{r}_1+\vec{r}_2}{2}\right)^2\right)$$

$$= \frac{1}{2}(\vec{r}_1-\vec{r}_2)^2+\frac{2}{3}\left(\vec{r}_3-\frac{\vec{r}_1+\vec{r}_2}{2}\right)^2 \qquad (3.14)$$

But,

$$\vec{r}_3-\frac{\vec{r}_1+\vec{r}_2}{2} = \vec{r}_3-\vec{r}_1+\frac{\vec{r}_1-\vec{r}_2}{2} = \vec{s}+\frac{\vec{r}}{2}, \text{ where } \vec{s} = \vec{r}_3-\vec{r}_1 \text{ and } \vec{r} = \vec{r}_1-\vec{r}_2.$$

Now the normalized wave function can be written as

$$\Psi(123) = \frac{1}{27^{1/4}L^{3/2}}\left(\frac{\beta}{\sqrt{\pi}}\right)^3 e^{-\frac{\beta^2}{2}\left[\frac{r^2}{2}+\frac{2}{3}\left(\vec{s}+\frac{\vec{r}}{2}\right)^2\right]} e^{i\vec{K}.\vec{R}} \psi_{spin}(123) \qquad (3.15)$$

Now, let us examine the spin part of $\Psi(123)$. This is important because the interaction between the nucleons is spin-dependent. The total wave function of the two protons must be anti-symmetric. We can see from eq. (3.15) that the space part of the wave function of the two protons is symmetric. This means that, for the total wave function of the two protons to be anti-symmetric, the spin part of the wave function of them must be anti-symmetric (singlet state) and thus

$$\psi_{spin}(12) = \psi_{00}(12) = \frac{1}{\sqrt{2}}\{\chi_+(1)\chi_-(2) - \chi_-(1)\chi_+(2)\} \qquad (3.16)$$

where $\chi_+(1)$ means that the first proton is spin up, and $\chi_-(2)$ means that the second proton is spin down. As the neutron can be spin up or spin down, here we choose it to be up, and so the spin wave function will be

$$\psi_{spin}(123) = \frac{1}{\sqrt{2}}\{\chi_+(1)\chi_-(2) - \chi_-(1)\chi_+(2)\}\chi_+(3) \qquad (3.17)$$

Whereas the spin wave function of the two protons is always anti-symmetric so that the two protons always interact with each other in the singlet state, the interaction between a proton and a neutron involves both the singlet and the triplet states. This can be seen as follows:

$$\psi_{spin}(123) = \frac{1}{\sqrt{2}}\{\chi_+(1)\chi_-(2)\chi_+(3) - \chi_-(1)\chi_+(2)\chi_+(3)\}$$

$$= \frac{1}{\sqrt{2}}\chi_+(1)\chi_-(2)\chi_+(3) - \frac{1}{\sqrt{2}}\,\chi_-(1)\chi_+(2)\chi_+(3)$$

$$= \frac{1}{\sqrt{2}}\chi_+(1)\chi_+(3)\chi_-(2) - \frac{1}{\sqrt{2}}\Big(\frac{1}{2}\chi_-(1)\chi_+(3) + \frac{1}{2}\chi_+(1)\chi_-(3)$$

$$+\frac{1}{2}\chi_-(1)\chi_+(3) - \frac{1}{2}\chi_+(1)\chi_-(3)\Big)\chi_+(2)$$

$$= \frac{1}{\sqrt{2}} \chi_+(1)\chi_+(3)\chi_-(2)$$

$$-\frac{1}{2}\left(\frac{1}{\sqrt{2}}[\chi_-(1)\chi_+(3) + \chi_+(1)\chi_-(3)]\right)\chi_+(2)$$

$$-\frac{1}{2}\left(\frac{1}{\sqrt{2}}[\chi_-(1)\chi_+(3) - \chi_+(1)\chi_-(3)]\right)\chi_+(2) \qquad (3.18)$$

We see that the interaction between the first proton and the neutron may be in the triplet or singlet states. The probability of it being in a triplet state $= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{2}\right)^2 = \frac{3}{4}$ and the probability of being in a singlet state $\left(\frac{-1}{2}\right)^2 = \frac{1}{4}$. The same thing can be said about the spin wave function of the second proton and the neutron.

3.5 Binding Energy of ^{3}He Using Harmonic Oscillator Nuclear Shell Model:

The binding energy of a ^{3}He nucleus is the energy needed to split it to its components which are two protons and one neutron. In order to find the binding energy of ^{3}He we will use the Skyrme interactions (see eq. 2.14) to represent the interaction between nucleons within the harmonic oscillator potential well. The Hamiltonian of the system can be written as

$$H = \sum_{i=1}^{3} \frac{-\hbar^2}{2m} \nabla_i^2 + v_{12} + v_{13} + v_{23} + v_{123}$$

$$= \frac{-\hbar^2}{2m} [\nabla_1^2 + \nabla_2^2 + \nabla_3^2] + v_{12} + v_{13} + v_{23} + v_{123} \qquad (3.19)$$

where $\sum_{i=1}^{3} \frac{-\hbar^2}{2m} \nabla_i^2$ represents the kinetic energy of the three nucleons. The terms v_{12}, v_{13}, v_{23} and v_{123} represent the two- and the three-body interactions between the three nucleons. We will use the Skyrme interactions (see eq. (2.14)) to represent these terms. The expectation value of the Hamiltonian gives the total energy

$$E_{total} = \langle \Psi | H | \Psi \rangle$$

$$= \langle \Psi | \frac{-\hbar^2}{2m} [\nabla_1^2 + \nabla_2^2 + \nabla_3^2] + v_{12} + v_{13} + v_{23} + v_{123} | \Psi \rangle$$

$$= \langle \Psi | \frac{-\hbar^2}{2m} [\nabla_1^2 + \nabla_2^2 + \nabla_3^2] | \Psi \rangle + \langle \Psi | v_{12} | \Psi \rangle + \langle \Psi | v_{13} | \Psi \rangle + \langle \Psi | v_{23} | \Psi \rangle +$$

$$\langle \Psi | v_{123} | \Psi \rangle \qquad (3.20)$$

We will now evaluate the first term in eq. (3.20) which represents the kinetic energy contribution to the total energy. For simplicity, we will do the calculations in the x-direction. The space part of the wave function in (3.15) can be written as:

$$\Psi_x(123) = \frac{A_x}{L^{3/2}} e^{-\frac{\beta^2}{3}(x^2 + s_x^2 + x s_x)} e^{i K_x X},$$

where

$$x = x_1 - x_2, s_x = x_3 - x_1, \text{ and } X = \frac{x_1 + x_2 + x_3}{3}$$

Then,

$$\frac{\partial^2 \Psi_x}{\partial x_1^2} = \frac{\partial^2 \Psi_x}{\partial x^2} + \frac{\partial^2 \Psi_x}{\partial s_x^2} + \frac{1}{9} \frac{\partial^2 \Psi_x}{\partial X^2} + \frac{2}{3} \frac{\partial^2 \Psi_x}{\partial x \partial X} - \frac{2}{3} \frac{\partial^2 \Psi_x}{\partial s_x \partial X} - 2 \frac{\partial^2 \Psi_x}{\partial x \partial s_x} \qquad (3.21)$$

$$\frac{\partial^2\Psi_x}{\partial x_2^2} = \frac{\partial^2\Psi_x}{\partial x^2} + \frac{1}{9}\frac{\partial^2\Psi_x}{\partial X^2} - \frac{2}{3}\frac{\partial^2\Psi_x}{\partial x\partial X} \tag{3.22}$$

$$\frac{\partial^2\Psi_x}{\partial x_3^2} = \frac{\partial^2\Psi_x}{\partial s_x^2} + \frac{1}{9}\frac{\partial^2\Psi_x}{\partial X^2} + \frac{2}{3}\frac{\partial^2\Psi_x}{\partial s_x\partial X} \tag{3.23}$$

Now, by adding (3.21), (3.22), and (3.23)

$$\frac{\partial^2\Psi_x}{\partial x_1^2} + \frac{\partial^2\Psi_x}{\partial x_2^2} + \frac{\partial^2\Psi_x}{\partial x_3^2} = 2\left(\frac{\partial^2\Psi_x}{\partial x^2} + \frac{\partial^2\Psi_x}{\partial s_x^2} - \frac{\partial^2\Psi_x}{\partial x\partial s_x}\right) + \frac{1}{3}\frac{\partial^2\Psi_x}{\partial X^2} \tag{3.24}$$

In order to evaluate eq. (3.24) we will find

$$\frac{\partial^2\Psi_x}{\partial x^2} = \left(\frac{4}{9}\beta^4x^2 + \frac{1}{9}\beta^4 s_x^2 + \frac{4}{9}\beta^4 x s_x - \frac{2}{3}\beta^2\right)\Psi_x \tag{3.25}$$

$$\frac{\partial^2\Psi_x}{\partial s_x^2} = \left(\frac{1}{9}\beta^4x^2 + \frac{4}{9}\beta^4 s_x^2 + \frac{4}{9}\beta^4 x s_x - \frac{2}{3}\beta^2\right)\Psi_x \tag{3.26}$$

$$\frac{\partial^2\Psi_x}{\partial x\partial s_x} = \left(\frac{2}{9}\beta^4x^2 + \frac{2}{9}\beta^4 s_x^2 + \frac{5}{9}\beta^4 x s_x - \frac{1}{3}\beta^2\right)\Psi_x \tag{3.27}$$

Then eq. (3.24) gives

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi_x}{\partial x_1^2} + \frac{\partial^2\Psi_x}{\partial x_2^2} + \frac{\partial^2\Psi_x}{\partial x_3^2}\right) = -\frac{\hbar^2}{2m}\left[\frac{2}{3}\beta^4(x^2 + s_x^2 + x s_x) - 2\beta^2\right]\Psi_x + \frac{\hbar^2 K_x^2}{2M}\Psi_x$$

where $M = 3m$. The first part gives the internal kinetic energy of the nucleons and the last term gives the kinetic energy of the CM. Thus

$$\langle T_x\rangle = \langle\Psi_x|\left\{-\frac{\hbar^2}{2m}\left[\frac{2}{3}\beta^4(x^2 + s_x^2 + x s_x) - 2\beta^2\right] + \frac{\hbar^2 K_x^2}{2M}\right\}|\Psi_x\rangle$$

Since

$$\int e^{-\frac{2}{3}\beta^2(x^2 + s_x^2 + x s_x)} ds_x dx = f(\beta^2) = \frac{1}{|A_x^2|} = \frac{\sqrt{3}}{2}\frac{\pi}{\beta^2}$$

and

$$\int (x^2 + s_x^2 + xs_x) e^{-\frac{2}{3}\beta^2 (x^2 + s_x^2 + xs_x)} ds_x dx = -\frac{3}{2} \frac{df(\beta^2)}{d\beta^2}$$

After some algebra, the kinetic energy can be expressed as

$$\langle T_x \rangle = \frac{\hbar^2 \beta^2}{2m} + \frac{\hbar^2 K_x^2}{2M}$$

Adding the kinetic energy from the y and z components of the wave function we get for the total kinetic energy:

$$\langle T \rangle = \frac{3}{2} \frac{\hbar^2 \beta^2}{m} + \frac{\hbar^2 K^2}{2M} \tag{3.28}$$

Now let us evaluate $\langle \Psi | v_{12} | \Psi \rangle$, $\langle \Psi | v_{13} | \Psi \rangle$, $\langle \Psi | v_{23} | \Psi \rangle$ and $\langle \Psi | v_{123} | \Psi \rangle$ to find the potential energy contribution.

$$\langle \Psi | v_{12} | \Psi \rangle = -\int \Psi^* t_0 \big(1 + x_0 P_\sigma(12)\big) \delta(\vec{r}_1 - \vec{r}_2) \Psi \, d^3 r_1 d^3 r_2 d^3 r_3$$

but it is clear from (3.16) that the two protons form a singlet state, thus the operator P_σ gives -1 and so

$$\begin{aligned}
\langle \Psi | v_{12} | \Psi \rangle &= -(1 - x_0) t_0 \frac{1}{27^{1/2} L^3} \left(\frac{\beta}{\sqrt{\pi}} \right)^6 \int e^{-\beta^2 \left[\frac{r^2}{2} + \frac{2}{3} \left(\vec{s} + \frac{\vec{r}}{2} \right)^2 \right]} \delta(\vec{r}) d^3 r d^3 s d^3 R \\
&= -(1 - x_0) \frac{1}{27^{1/2}} \left(\frac{\beta}{\sqrt{\pi}} \right)^6 \int e^{-\frac{2}{3} \beta^2 s^2} d^3 s \\
&= -\frac{1}{2\sqrt{2}} \frac{1}{\pi^{3/2}} \beta^3 (1 - x_0) t_0
\end{aligned}$$

Using the same procedure, we can evaluate $\langle \Psi | v_{13} | \Psi \rangle$ and $\langle \Psi | v_{23} | \Psi \rangle$. From (3.18) we note that a neutron-proton pair is $3/4$ of the time in a triplet state and $1/4$ of the time in a singlet state. When the operator P_σ acts on the triplet wave function we get $+1$ and on a singlet state we get -1 so

$$\langle\Psi|v_{13}|\Psi\rangle = \langle\Psi|v_{23}|\Psi\rangle = -\frac{1}{2\sqrt{2}}\frac{1}{\pi^{3/2}}\beta^3\left(\frac{3}{4}(1+x_0)+\frac{1}{4}(1-x_0)\right)$$

$$= -\frac{1}{2\sqrt{2}}\frac{1}{\pi^{3/2}}\beta^3\left(1+\frac{x_0}{2}\right)$$

$$\langle\Psi|v_{12}|\Psi\rangle + \langle\Psi|v_{13}|\Psi\rangle + \langle\Psi|v_{23}|\Psi\rangle = -\frac{3}{2\sqrt{2}}\frac{t_0}{\pi^{3/2}}\beta^3 \qquad (3.29)$$

For the three-body Skyrme interaction

$$\langle\Psi|v_{123}|\Psi\rangle = \int \Psi^* t_3 \delta(\vec{r}_1-\vec{r}_2)\delta(\vec{r}_2-\vec{r}_3)\Psi \, d^3r_1 d^3r_2 d^3r_3$$

$$= t_3\frac{1}{27^{1/2}L^3}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \int e^{-\beta^2\left[\frac{r^2}{2}+\frac{2}{3}\left(\vec{s}+\frac{\vec{r}}{2}\right)^2\right]}\delta(\vec{r})\delta\big(-(\vec{r}+\vec{s})\big)d^3r d^3s d^3R$$

$$= t_3\frac{1}{27^{1/2}}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \int e^{-\beta^2\left[\frac{r^2}{2}+\frac{2}{3}\left(-\vec{r}+\frac{\vec{r}}{2}\right)^2\right]}\delta(\vec{r})d^3r$$

$$= \frac{t_3}{\sqrt{27}}\frac{\beta^6}{\pi^3} \qquad (3.30)$$

Thus,

$$E_{total} = \frac{\hbar^2K^2}{2M}+\frac{3}{2}\frac{\hbar^2\beta^2}{m}-\frac{3}{2\sqrt{2}}\frac{t_0}{\pi^{3/2}}\beta^3+\frac{1}{\sqrt{27}}\frac{t_3}{\pi^3}\beta^6 = \frac{\hbar^2K^2}{2M}+E_{int} \qquad (3.31)$$

where the term $\frac{\hbar^2K^2}{2M}$ represents the kinetic energy of the nucleus itself which does not contribute to the internal energy of the nucleus. Thus the binding energy of ^{3}He nucleus can be written as

$$B(\beta) = -E_{int} = -\frac{3}{2}\frac{\hbar^2\beta^2}{m}+\frac{3}{2\sqrt{2}}\frac{t_0}{\pi^{3/2}}\beta^3-\frac{1}{\sqrt{27}}\frac{t_3}{\pi^3}\beta^6 \qquad (3.32)$$

Now, let us evaluate β by using the experimental value of the rms radius of the [3]He nucleus which is 1.76 fm (see Table 3). We will use the wave function of the individual nucleon as the radius is similar for the three nucleons, so

$$\langle r^2\rangle = \int \psi^*(\vec{r}_i) r^2 \psi(\vec{r}_i) d^3r$$

$$1.76^2 = 4\pi \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \int r^4\, e^{-\beta^2 r^2} dr = \frac{3}{2}\frac{1}{\beta^2}$$

$$\Longrightarrow \beta = 0.696\ fm^{-1}$$

Note that $E_{int}(\beta)$ has the general form:

$$E_{int}(\beta) = a\,\beta^2 - ct_0\beta^3 + \ gt_3\,\beta^6$$

where

$a = \frac{3}{2}\frac{\hbar^2}{m} = 61.9\ MeV\ fm^2$, $c = \frac{3}{2\sqrt{2}}\frac{1}{\pi^{3/2}} = 0.19$, and $g = \frac{1}{\sqrt{27}}\frac{1}{\pi^3} = 6.2 \times 10^{-3}$.

Also note that c and g are dimensionless.

Now we want to find the values of t_0 and t_3 which satisfy the following two conditions: the first condition is that in the ground state the energy is $-B_0 = -7.718$ MeV thus

$$a\,\beta^2 - ct_0\beta^3 + gt_3\beta^6|_{\beta=0.696} = -\,B_0 \tag{3.33}$$

and the second one is the equilibrium condition which means that the energy is minimum (the binding energy is maximum) at this β so we have

$$\left.\frac{\partial E_{int}}{\partial \beta}\right|_{\beta=0.696} = 0,$$

thus

$$a(2\beta) - c(3\beta^2)t_0 + g(6\beta^5)t_3|_{\beta=0.696} = 0 \tag{3.34}$$

By solving these two equations (3.33) and (3.34) we get:

$$t_0 = \frac{4a}{3c}\frac{1}{\beta} + \frac{2B_0}{c}\frac{1}{\beta^3} \quad \text{and} \quad t_3 = \frac{a}{3gs}\frac{1}{\beta^4} + \frac{B_0}{g}\frac{1}{\beta^6} \tag{3.35}$$

After substituting the values of a , c, g and B_0, in (3.35) we get

$$t_0 = 865\ \ MeV\ fm^3 \text{ and } t_3 = 25152\ MeV\ fm^6. \tag{3.36}$$

In Fig. (3.1) the energy of the ^{3}He nucleus is plotted as a function of β. It is clear that at $\beta = 0.696$ the energy has its minimum (-7.718 MeV).

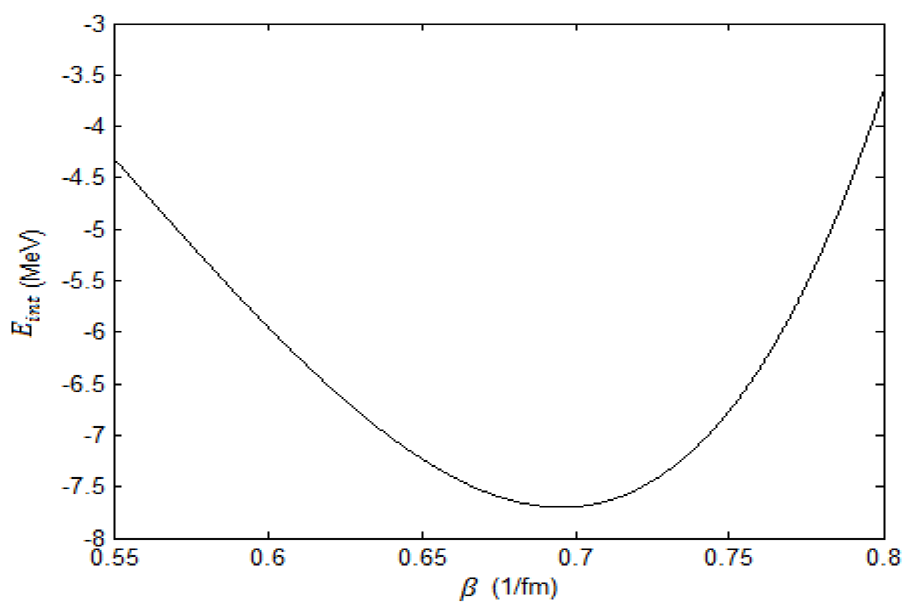


Fig. 3.1. The energy of ^{3}He nucleus as a function of β

CHAPTER 4. MEDIUM DEPENDENCE OF THE BINDING ENERGY

In the previous chapter we derived the formula of the binding energy of the ^{3}He nucleus when it is alone. Now let us consider ^{3}He nuclei moving in a hot low-density vapor of protons and neutrons which is the problem of our research. We will assume thermal and chemical equilibrium between the ^{3}He nuclei and the surrounding vapor.

4.1 WAVE FUNCTION OF ^{3}He –NEUTRON SYSTEM:

For simplicity, let us at first derive the wave function of a system composed only of one ^{3}He nucleus and one neutron confined in a cubical box of length L. We will treat this neutron as a free particle and so its wave function can be written as

$$\frac{1}{L^{3/2}} e^{i\vec{k}.\vec{r}_4} \tag{4.1}$$

where $\vec{k}$ is the wave vector of the free neutron and $\vec{r}_4$ is its position vector. In eq. (4.1) we assumed that the neutron has one value of linear momentum for simplicity. But as we mentioned above there is thermal equilibrium between ^{3}He nuclei and the surrounding vapor which means that we must take the thermal average over all values of momentum as we will see in the next chapter. Now, using the same change of variables as in eq. (3.14) the space part of wave function of this system can be written as

$$\psi_{space}(1234) = \frac{1}{27^{1/4}L^{3/2}} \left(\frac{\beta}{\sqrt{\pi}}\right)^3 e^{-\frac{\beta^2}{2}\left[\frac{r^2}{2} + \frac{2}{3}(\vec{r}_3 - \vec{\rho})^2\right]} e^{i\vec{K}.\vec{R}} \frac{1}{L^{3/2}} e^{i\vec{k}.\vec{r}_4} \tag{4.2}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$, $\vec{\rho} = \frac{\vec{r}_1+\vec{r}_2}{2}$ and $\vec{R} = \frac{\vec{r}_1+\vec{r}_2+\vec{r}_3}{3}$, so we can rewrite the wave function in eq. (4.2) to be

$$\psi_{space}(1234) = \frac{1}{27^{1/4}L^3}\left(\frac{\beta}{\sqrt{\pi}}\right)^3 e^{-\frac{\beta^2 r^2}{4}} e^{\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_3 - \vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_3} e^{i\vec{k}.\vec{r}_4} \quad (4.3)$$

But the total wave function,

$$\Psi_{total}(1234) = \psi_{space}(1234)\psi_{spin}(1234) \quad (4.4)$$

must be anti-symmetric under the exchange of any two identical nucleons as all the nucleons (nucleons within ^{3}He nucleus and the free neutron) are fermions. To be more clear let us write the wave function as

$$\Psi_{total}(1234) = \psi_{space}(12)\psi_{spin}(12)\psi_{space}(34)\psi_{spin}(34) \quad (4.5)$$

The total wave function of the two protons must be anti-symmetric. It is clear from eq. (4.2) that the space part of the wave function of the two protons is symmetric. This means that, for the total wave function of the two protons to be anti-symmetric, the spin part of the wave function of them must be anti-symmetric (singlet state) and thus

$$\psi_{spin}(12) = \psi_{00}(12) = \frac{1}{\sqrt{2}}\{\chi_+(1)\chi_-(2) - \chi_-(1)\chi_+(2)\} \quad (4.6)$$

Also, the total wave function of the two neutrons (the neutron confined in the ^{3}He nucleus and the free one) must be also anti-symmetric. Here we have two cases:

the first one is that when the space part of the wave function of the two neutrons is symmetric, and its spin part is then anti-symmetric, and is given by

$$\psi_{spin}(34) = \psi_{00}(34) = \frac{1}{\sqrt{2}}\{\chi_{+}(3)\chi_{-}(4) - \chi_{-}(3)\chi_{+}(4)\} \tag{4.7}$$

The second case is that when the space part of the two neutrons is anti-symmetric, the spin part of them must be symmetric. Thus, in this case we have triplet symmetric states of the spin wave function of the two neutrons

$$\psi_{10}(34) = \frac{1}{\sqrt{2}}\{\chi_{+}(3)\chi_{-}(4) + \chi_{-}(3)\chi_{+}(4)\} \tag{4.8.a}$$

$$\psi_{11}(34) = \chi_{+}(3)\chi_{+}(4) \tag{4.8.b}$$

$$\psi_{1,-1}(34) = \chi_{-}(3)\chi_{-}(4) \tag{4.8.c}$$

Now after the above discussion we can conclude that the total wave function can be written as

$$\Psi_{total}(1234) = \sqrt{\frac{3}{4}}\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti}(34)\psi_{spin}^{t}(34) + \sqrt{\frac{1}{4}}\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym}(34)\psi_{spin}^{s}(34) \tag{4.9}$$

$$\Psi_{total}(1234) = \sqrt{\frac{3}{4}}\,\frac{N}{27^{1/4}L^3}\left(\frac{\beta}{\sqrt{\pi}}\right)^3 e^{-\frac{\beta^2 r^2}{4}} e^{\frac{2}{3}i\vec{K}.\vec{\rho}} \left\{ e^{-\frac{\beta^2}{3}(\vec{r}_3 - \vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_3} e^{i\vec{k}.\vec{r}_4} - e^{-\frac{\beta^2}{3}(\vec{r}_4 - \vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_4} e^{i\vec{k}.\vec{r}_3} \right\} \psi^s_{spin}(12)\psi^t_{spin}(34)$$
$$+ \sqrt{\frac{1}{4}}\frac{N'}{27^{1/4}L^3}\left(\frac{\beta}{\sqrt{\pi}}\right)^3 e^{-\frac{\beta^2 r^2}{4}} e^{\frac{2}{3}i\vec{K}.\vec{\rho}} \left\{ e^{-\frac{\beta^2}{3}(\vec{r}_3 - \vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_3} e^{i\vec{k}.\vec{r}_4} + e^{-\frac{\beta^2}{3}(\vec{r}_4 - \vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_4} e^{i\vec{k}.\vec{r}_3} \right\} \psi^s_{spin}(12)\psi^s_{spin}(34) \quad (4.10)$$

where ψ^s_{spin} denotes the singlet symmetric spin state and ψ^t_{spin} the triplet anti-symmetric spin state. Note that the minus sign in the first part of the total wave function makes the space wave function anti-symmetric in the two neutrons (particles 3 and 4) while the plus sign in the second part makes it symmetric in the two neutrons. Also in eq. (4.10) N and N' are the normalization constants for the symmetric and antisymmetric space wave functions of the two neutrons (note that the spin part is already normalized). After applying the normalization condition we get

$$N = \frac{1}{\sqrt{2}\left\{1 - \left(\frac{\sqrt{6\pi}}{\beta L}\right)^3 e^{-\frac{3}{2\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}\right\}^{1/2}} \quad (4.11)$$

$$N' = \frac{1}{\sqrt{2}\left\{1 + \left(\frac{\sqrt{6\pi}}{\beta L}\right)^3 e^{-\frac{3}{2\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}\right\}^{1/2}} \quad (4.12)$$

The term $\left(\frac{\sqrt{6\pi}}{\beta L}\right)^3 e^{-\frac{3}{2\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}$ can be neglected as it is very small. This is because the size of the box (L) is much larger than the size of the ^{3}He nucleus which is of the order of few femtometers. To be more clear let us calculate the maximum value of this term. If we consider a box of edge $L = 30\ fm$ and using the value of $\beta = 0.696\ fm^{-1}$ which was found in the previous chapter the maximum value of this term will be 8.99×10^{-3} which is $\ll 1$. Thus, we get

$$N = N' \cong \frac{1}{\sqrt{2}} \tag{4.13}$$

4.2 Binding Energy of ^{3}He –Neutron System:

Now let us find the binding energy of a ^{3}He nucleus when it is immersed in a vapor of nucleons. The Hamiltonian of the ^{3}He-neutron system can be written as

$$H = \sum_{i=1}^{3} \frac{-\hbar^2}{2m} \nabla_i^2 + v_{12} + v_{13} + v_{23} + v_{123} + \frac{-\hbar^2}{2m} \nabla_4^2 + v_{14} + v_{24} + v_{34} + v_{124} + v_{134} + v_{234} \tag{4.14}$$

where $\sum_{i=1}^{3} \frac{-\hbar^2}{2m} \nabla_i^2$ and $\frac{-\hbar^2}{2m} \nabla_4^2$ represent the kinetic energies of the four nucleons. The terms v_{12}, v_{13}, v_{23} and v_{123} represent the two-body interactions and the three-body interactions between the bound nucleons while the terms v_{14}, v_{24}, v_{34}, v_{124}, v_{134}, and v_{234} represent the two-body interactions and the three-body interactions between the free neutron and the nucleons bound in the ^{3}He nucleus. We will use the Skyrme interactions defined in eq. (2.14) to represent these terms.

As we have shown in chapter 3, the Hamiltonian of the ^{3}He nucleus can be separated into the Hamiltonian of the center-of-mass H_{CM} and the internal Hamiltonian H_{int} which stands for the motion of the nucleons within the ^{3}He nucleus (this is clear in eq. (3.7)). This means that the Hamiltonian of the system can be written as

$$H = H_{int}(123) + H_{CM}(123) + \frac{-\hbar^2}{2m}\nabla_4^2 +$$
$$v_{14} + v_{24} + v_{34} + v_{124} + v_{134} + v_{234} \qquad (4.15.a)$$

Or

$$= H_{int}(124) + H_{CM}(124) + \frac{-\hbar^2}{2m}\nabla_3^2 +$$
$$v_{13} + v_{23} + v_{43} + v_{123} + v_{143} + v_{243} \qquad (4.15.b)$$

Eqs. (4.15.a) and (4.15.b) are equivalent. The only difference between them is the exchange between the two neutrons (particle 3 and particle 4) which reflects the indistinguishability between these two particles.

The expectation value of the Hamiltonian gives the total energy of the ^{3}He-neutron system:

$$E_{total} = \langle \Psi_{total} | H | \Psi_{total} \rangle \qquad (4.16)$$

$$= \langle \Psi_{total} | H_{int} + H_{CM} + \frac{-\hbar^2}{2m}\nabla_4^2 + v_{14} + v_{24} + v_{34} + v_{124} + v_{134} + v_{234} | \Psi_{total} \rangle$$

$$= \langle \Psi_{total} | H_{int} | \Psi_{total} \rangle + \langle \Psi_{total} | H_{CM} | \Psi_{total} \rangle + \langle \Psi_{total} | \frac{-\hbar^2}{2m}\nabla_4^2 | \Psi_{total} \rangle$$
$$+ \langle \Psi_{total} | v_{14} + v_{24} + v_{34} + v_{124} + v_{134} + v_{234} | \Psi_{total} \rangle \qquad (4.17)$$

But as motioned above the total wave function has two parts: symmetric and anti-symmetric as we can see from eq. (4.9). These two parts results from the exchange of the two identical neutrons (particle 3 and particle 4). This means that when we want to evaluate eq. (4.17) we must be careful to use the two forms of the Hamiltonian as it is defined in eqs. (4.15.a) and (4.15.b) to stand for this exchange. In the last term of eq. (4.17) we wrote v_{14}, v_{24}, v_{34}, v_{124}, v_{134}, and v_{234}, but as we will see below these terms will be v_{13}, v_{23}, v_{43}, v_{123}, v_{143}, and v_{243} when we exchange the two neutrons in the wave function. We will show this explicitly in evaluating each term in eq. (4.17) below. The first three terms in the above equation can be evaluated using the same procedure. Here we will evaluate the first one explicitly

$$\langle H_{int}\rangle = \langle \Psi_{total}|H_{int}|\Psi_{total}\rangle$$

Using eq. (4.9) we get

$$\langle H_{int}\rangle = \langle \sqrt{\frac{3}{4}}\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti*}(34)\psi_{spin}^{t}(34)$$

$$+\sqrt{\frac{1}{4}}\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym*}(34)\psi_{spin}^{s}(34)|H_{int}|$$

$$\sqrt{\frac{3}{4}}\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti}(34)\psi_{spin}^{t}(34)$$

$$+\sqrt{\frac{1}{4}}\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym}(34)\psi_{spin}^{s}(34)\rangle$$

$$\langle H_{int}\rangle = \frac{3}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti*}(34)\psi_{spin}^{t}(34)|H_{int}|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti}(34)\psi_{spin}^{t}(34)\rangle$$

$$+\frac{1}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym*}(34)\psi_{spin}^{s}(34)|H_{int}|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym}(34)\psi_{spin}^{s}(34)\rangle$$

$$+\frac{\sqrt{3}}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti*}(34)\psi_{spin}^{t}(34)|H_{int}|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym}(34)\psi_{spin}^{s}(34)\rangle$$

$$+\frac{\sqrt{3}}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym*}(34)\psi_{spin}^{s}(34)|H_{int}|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti}(34)\psi_{spin}^{t}(34)\rangle$$

We can notice that the last two terms will vanish because we have orthogonal states; both the space parts and the spin parts of the bra and ket wave functions are orthogonal to each other. Thus, we need to evaluate the first two integrals only

$$\langle H_{int}\rangle = \frac{3}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti*}(34)\psi_{spin}^{t}(34)|H_{int}|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti}(34)\psi_{spin}^{t}(34)\rangle$$

$$+\frac{1}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym*}(34)\psi_{spin}^{s}(34)|H_{int}|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym}(34)\psi_{spin}^{s}(34)\rangle \tag{4.18}$$

Since the spin wave functions are normalized we are left with evaluating the space integrals, and so eq. (4.18) will be

$$\langle H_{int}\rangle = \frac{3}{4}\int \psi_{space}^{sym*}(12)\psi_{space}^{anti*}(34)H_{int}$$

$$\psi_{space}^{sym}(12)\psi_{space}^{anti}(34)d^3rd^3\rho d^3r_3d^3r_4$$

$$+\frac{1}{4}\int \psi_{space}^{sym*}(12)\psi_{space}^{sym*}(34)H_{int}$$

$$\psi_{space}^{sym}(12)\psi_{space}^{sym}(34)d^3rd^3\rho d^3r_3d^3r_4 \qquad (4.19)$$

It is clear that the first integral in the above equation contains the anti-symmetric space wave function of the two neutrons while the second one contains the symmetric space wave function of the two neutrons. Let us now evaluate the first integral which gives

$$= \frac{3}{4}\frac{N^2}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \left\{\int e^{-\frac{\beta^2r^2}{4}} e^{-\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{-i\frac{\vec{K}}{3}.\vec{r}_3} e^{-i\vec{k}.\vec{r}_4} H_{int}(123)\right.$$

$$e^{-\frac{\beta^2r^2}{4}} e^{\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_3} e^{i\vec{k}.\vec{r}_4} d^3rd^3\rho d^3r_3d^3r_4$$

$$+\int e^{-\frac{\beta^2r^2}{4}} e^{-\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_4-\vec{\rho})^2} e^{-i\frac{\vec{K}}{3}.\vec{r}_4} e^{-i\vec{k}.\vec{r}_3} H_{int}(124)$$

$$e^{-\frac{\beta^2r^2}{4}} e^{\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_4-\vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_4} e^{i\vec{k}.\vec{r}_3} d^3rd^3\rho d^3r_3d^3r_4$$

$$-\int e^{-\frac{\beta^2r^2}{4}} e^{-\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{-i\frac{\vec{K}}{3}.\vec{r}_3} e^{-i\vec{k}.\vec{r}_4} H_{int}(123)$$

$$e^{-\frac{\beta^2r^2}{4}} e^{\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_4-\vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_4} e^{i\vec{k}.\vec{r}_3} d^3rd^3\rho d^3r_3d^3r_4$$

$$-\int e^{-\frac{\beta^2r^2}{4}} e^{-\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_4-\vec{\rho})^2} e^{-i\frac{\vec{K}}{3}.\vec{r}_4} e^{-i\vec{k}.\vec{r}_3} H_{int}(124)$$

$$\left. e^{-\frac{\beta^2r^2}{4}} e^{\frac{2}{3}i\vec{K}.\vec{\rho}} e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{i\frac{\vec{K}}{3}.\vec{r}_3} e^{i\vec{k}.\vec{r}_4} d^3rd^3\rho d^3r_3d^3r_4\right\}$$

Note from the above equation that we used both $H_{int}(123)$ and $H_{int}(124)$ to fit the exchange between the two neutrons in the wave function. But we know that the when the Hamiltonian operator $H_{int}(123)$ or $H_{int}(124)$ acts on the wave function it gives $-B_0$, thus we get

$$= -\frac{3}{4}\frac{N^2}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 B_0\left\{\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{2}{3}\beta^2(\vec{r}_3-\vec{\rho})^2}\, d^3r d^3\rho d^3r_3 d^3r_4\right.$$

$$+\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{2}{3}\beta^2(\vec{r}_4-\vec{\rho})^2}\, d^3r d^3\rho d^3r_3 d^3r_4$$

$$-\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)} d^3r d^3\rho d^3r_3 d^3r_4$$

$$\left.-\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_3-\vec{r}_4)} d^3r d^3\rho d^3r_3 d^3r_4\right\}$$

Here we can see that the first two integrals are equal and the last two integrals are equal, and so we have

$$= -\frac{3}{4}\frac{2N^2}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 B_0\left\{\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{2}{3}\beta^2(\vec{r}_3-\vec{\rho})^2}\, d^3r d^3\rho d^3r_3 d^3r_4\right.$$

$$\left.-\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)} d^3r d^3\rho d^3r_3 d^3r_4\right\}$$

By evaluating the above two integrals we get

$$= -\frac{3}{4}\frac{2N^2}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 B_0\left\{\sqrt{27}L^6\left(\frac{\sqrt{\pi}}{\beta}\right)^6\frac{1}{2N^2}\right\} = -\frac{3}{4}B_0$$

The second integral for the symmetric space wave function part in eq. (4.19) can be evaluated using the same procedure, and the result will be similar except for a plus sign instead of the minus sign so eq. (4.19) can be written as

$$\langle H_{int}\rangle = -\frac{3}{4}B_0 - \frac{1}{4}B_0 = -B_0$$

Using the same technique for the second two terms we get $\langle H_{CM}\rangle = \frac{\hbar^2 K^2}{2M}$ and $\langle \frac{-\hbar^2}{2m}\nabla_4^2\rangle = \frac{\hbar^2 k^2}{2m}$.

Thus, eq. (4.17) will be

$$= -B_0 + \frac{\hbar^2 K^2}{2M} + \frac{\hbar^2 k^2}{2m} + \langle \Psi_{total}|\, v_{14} + v_{24} + v_{34} |\Psi_{total}\rangle$$
$$+\langle \Psi_{total}| v_{124} + v_{134} + v_{234} |\Psi_{total}\rangle \qquad (4.20)$$

where B_0 represents the binding energy of an isolated ^{3}He nucleus (see Table 3). The terms $\frac{\hbar^2 K^2}{2M}$ and $\frac{\hbar^2 k^2}{2m}$ represent the kinetic energies of the ^{3}He nucleus and the free neutron respectively. The terms $\langle \Psi_{total}|\, v_{14} + v_{24} + v_{34} |\Psi_{total}\rangle$ and $\langle \Psi_{total}| v_{124} + v_{134} + v_{234} |\Psi_{total}\rangle$ stand for the potential energy contribution which results from the two- and three-body interactions between the free neutron and the bound nucleons inside the ^{3}He nucleus. Again we will use the Skyrme interaction to represent these interactions.

$$\langle \Psi_{total}|\, v_{14} + v_{24} + v_{34} |\Psi_{total}\rangle = \langle \Psi_{total}| v_{14} |\Psi_{total}\rangle + \langle \Psi_{total}| v_{24} |\Psi_{total}\rangle$$
$$+\langle \Psi_{total}| v_{34} |\Psi_{total}\rangle \qquad (4.21)$$

The first term is

$\langle v_{14} \rangle = \langle \Psi_{total} | v_{14} | \Psi_{total} \rangle$

Using the same procedure in deriving eq. (4.18) we get

$$\langle v_{14} \rangle = \frac{3}{4} \langle \psi_{space}^{sym*}(12) \psi_{spin}^{s}(12) \psi_{space}^{anti*}(34) \psi_{spin}^{t}(34) |\ v_{14} |$$

$$\psi_{space}^{sym}(12) \psi_{spin}^{s}(12) \psi_{space}^{anti}(34) \psi_{spin}^{t}(34) \rangle$$

$$+ \frac{1}{4} \langle \psi_{space}^{sym*}(12) \psi_{spin}^{s}(12) \psi_{space}^{sym*}(34) \psi_{spin}^{s}(34) |\ v_{14} |$$

$$\psi_{space}^{sym}(12) \psi_{spin}^{s}(12) \psi_{space}^{sym}(34) \psi_{spin}^{s}(34) \rangle$$

$$= \frac{3}{4} \langle \psi_{space}^{sym*}(12) \psi_{spin}^{s}(12) \psi_{space}^{anti*}(34) \psi_{spin}^{t}(34) \left| -t_0 \big(1 + x_0 P_\sigma(14)\big) \delta(\vec{r}_1 - \vec{r}_4) \right|$$

$$\psi_{space}^{sym}(12) \psi_{spin}^{s}(12) \psi_{space}^{anti}(34) \psi_{spin}^{t}(34) \rangle$$

$$+ \frac{1}{4} \langle \psi_{space}^{sym*}(12) \psi_{spin}^{s}(12) \psi_{space}^{sym*}(34) \psi_{spin}^{s}(34) \left| -t_0 \big(1 + x_0 P_\sigma(14)\big) \delta(\vec{r}_1 - \vec{r}_4) \right|$$

$$\psi_{space}^{sym}(12) \psi_{spin}^{s}(12) \psi_{space}^{sym}(34) \psi_{spin}^{s}(34) \rangle$$

$P_\sigma(14)$ here exchanges the spins of the first proton (particle 1) and the free neutron (particle 4) which may interact via the triplet or singlet interaction. The probability of this interaction to be in a triplet state is $3/4$ and the probability of it being in a singlet state is $1/4$. Thus, the operator $P_\sigma(14)$ will give

$$\frac{3}{4}(1 + x_0) + \frac{1}{4}(1 - x_0) = \left(1 + \frac{x_0}{2}\right)$$

Since the spin wave functions are normalized we are left with evaluating the space integrals, and so $\langle v_{14} \rangle$ will be

$$\langle v_{14}\rangle = -\frac{3}{4}t_0\left(1+\frac{x_0}{2}\right)\int \psi_{space}^{sym*}(12)\psi_{space}^{anti*}(34)\delta(\vec{r}_1-\vec{r}_4)$$

$$\psi_{space}^{sym}(12)\psi_{space}^{anti}(34)d^3r d^3\rho d^3r_3 d^3r_4$$

$$-\frac{1}{4}t_0\left(1+\frac{x_0}{2}\right)\int \psi_{space}^{sym*}(12)\psi_{space}^{sym*}(34)\delta(\vec{r}_1-\vec{r}_4)$$

$$\psi_{space}^{sym}(12)\psi_{space}^{sym}(34)d^3r d^3\rho d^3r_3 d^3r_4 \quad (4.22)$$

By evaluating the first integral in which the space wave function of the two neutrons is antisymmetric we get

$$-\frac{3}{4}t_0\left(1+\frac{x_0}{2}\right)\frac{N^2}{\sqrt{27L^6}}\left(\frac{\beta}{\sqrt{\pi}}\right)^6\left\{\int e^{-\frac{\beta^2r^2}{4}}e^{-\frac{2}{3}i\vec{K}.\vec{\rho}}e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2}e^{-i\frac{\vec{K}}{3}.\vec{r}_3}e^{-i\vec{k}.\vec{r}_4}\delta(\vec{r}_1-\vec{r}_4)\right.$$

$$e^{-\frac{\beta^2r^2}{4}}e^{\frac{2}{3}i\vec{K}.\vec{\rho}}e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2}e^{i\frac{\vec{K}}{3}.\vec{r}_3}e^{i\vec{k}.\vec{r}_4}d^3r d^3\rho d^3r_3 d^3r_4$$

$$+\int e^{-\frac{\beta^2r^2}{4}}e^{-\frac{2}{3}i\vec{K}.\vec{\rho}}e^{-\frac{\beta^2}{3}(\vec{r}_4-\vec{\rho})^2}e^{-i\frac{\vec{K}}{3}.\vec{r}_4}e^{-i\vec{k}.\vec{r}_3}\delta(\vec{r}_1-\vec{r}_3)$$

$$e^{-\frac{\beta^2r^2}{4}}e^{\frac{2}{3}i\vec{K}.\vec{\rho}}e^{-\frac{\beta^2}{3}(\vec{r}_4-\vec{\rho})^2}e^{i\frac{\vec{K}}{3}.\vec{r}_4}e^{i\vec{k}.\vec{r}_3}d^3r d^3\rho d^3r_3 d^3r_4$$

$$-\int e^{-\frac{\beta^2r^2}{4}}e^{-\frac{2}{3}i\vec{K}.\vec{\rho}}e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2}e^{-i\frac{\vec{K}}{3}.\vec{r}_3}e^{-i\vec{k}.\vec{r}_4}\delta(\vec{r}_1-\vec{r}_4)$$

$$e^{-\frac{\beta^2r^2}{4}}e^{\frac{2}{3}i\vec{K}.\vec{\rho}}e^{-\frac{\beta^2}{3}(\vec{r}_4-\vec{\rho})^2}e^{i\frac{\vec{K}}{3}.\vec{r}_4}e^{i\vec{k}.\vec{r}_3}d^3r d^3\rho d^3r_3 d^3r_4$$

$$-\int e^{-\frac{\beta^2r^2}{4}}e^{-\frac{2}{3}i\vec{K}.\vec{\rho}}e^{-\frac{\beta^2}{3}(\vec{r}_4-\vec{\rho})^2}e^{-i\frac{\vec{K}}{3}.\vec{r}_4}e^{-i\vec{k}.\vec{r}_3}\delta(\vec{r}_1-\vec{r}_3)$$

$$\left.e^{-\frac{\beta^2r^2}{4}}e^{\frac{2}{3}i\vec{K}.\vec{\rho}}e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2}e^{i\frac{\vec{K}}{3}.\vec{r}_3}e^{i\vec{k}.\vec{r}_4}d^3r d^3\rho d^3r_3 d^3r_4\right\}$$

The use of the two forms of the Hamiltonian: $H(123)$ and $H(124)$ is clear in the above equation. As we can see in the first integral we used $\delta(\vec{r}_1 - \vec{r}_4)$ which results from the use of the $H(123)$, while in the second integral we used $\delta(\vec{r}_1 - \vec{r}_3)$ which results from the use of the $H(124)$. This is because in the second integral we exchanged particle 3 and particle 4 (the two neutrons). The same thing can be said about the last two integrals.

$$= -\frac{3}{4}t_0\left(1+\frac{x_0}{2}\right)\frac{N^2}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6\Bigg\{\int e^{-\frac{\beta^2 r^2}{2}}e^{-\frac{2}{3}\beta^2(\vec{r}_3-\vec{\rho})^2}\delta(\vec{r}_1-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4$$

$$+\int e^{-\frac{\beta^2 r^2}{2}}e^{-\frac{2}{3}\beta^2(\vec{r}_4-\vec{\rho})^2}\delta(\vec{r}_1-\vec{r}_3)d^3r d^3\rho d^3r_3 d^3r_4$$

$$-\int e^{-\frac{\beta^2 r^2}{2}}e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2\right]}e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}\delta(\vec{r}_1-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4$$

$$-\int e^{-\frac{\beta^2 r^2}{2}}e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2\right]}e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_3-\vec{r}_4)}\delta(\vec{r}_1-\vec{r}_3)d^3r d^3\rho d^3r_3 d^3r_4\Bigg\}$$

Again we can see that the first two integrals are equal and the last two integrals are equal, and so we have

$$= -\frac{3}{4}t_0\left(1+\frac{x_0}{2}\right)\frac{2N^2}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6\Bigg\{\int e^{-\frac{\beta^2 r^2}{2}}e^{-\frac{2}{3}\beta^2(\vec{r}_3-\vec{\rho})^2}\delta(\vec{r}_1-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4$$

$$-\int e^{-\frac{\beta^2 r^2}{2}}e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2\right]}e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}\delta(\vec{r}_1-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4\Bigg\}$$

The integral $\int e^{-\frac{\beta^2 r^2}{2}}e^{-\frac{2}{3}\beta^2(\vec{r}_3-\vec{\rho})^2}\delta(\vec{r}_1-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4$ represents the interaction with the vapor (self-energy term). But when calculating the binding energy of a nucleus in a vapor we compare its energy with the energy of its nucleons when they are unbound but still in the vapor. Thus, in our work when we do such a comparison we notice that the interaction between the nucleons of ^{3}He nucleus with the surrounding nucleons in the vapor is approximately similar before and after the dissociation of the cluster. This means that the contribution from this term is very small and so it can be removed, thus we get

$$= -\frac{3}{4}t_0\left(1+\frac{x_0}{2}\right)\frac{2N^2}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \times$$

$$\left\{-\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2\right]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}\,\delta(\vec{r}_1-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4\right\}$$

The second integral for the symmetric space wave function part in eq. (4.22) can be evaluated using the same procedure, and the result will be similar except for a plus sign instead of the minus sign and so

$$\langle v_{14}\rangle = \left(3N^2-N'^2\right)\frac{t_0}{2}\left(1+\frac{x_0}{2}\right)\frac{1}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \times$$

$$\left\{\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2\right]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}\,\delta(\vec{r}_1-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4\right\} \quad (4.23)$$

Here we can notice the Pauli blocking effect in the term $e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}$. It is clear from this term that since the free neutron and the neutron confined within the ^{3}He cluster cannot have the same momentum (Pauli principle) the presence of this neutron around the ^{3}He cluster will affect its binding energy. The integral in eq. (4.23) can be evaluated as follows

$$\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)} \delta(\vec{r}_1-\vec{r}_4) d^3r d^3\rho d^3r_3 d^3r_4$$

But using $\vec{r}=\vec{r}_1-\vec{r}_2$ and $\vec{\rho}=\frac{\vec{r}_1+\vec{r}_2}{2}$ we get

$$= \int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)} \delta\left(\frac{1}{2}\vec{r}+\vec{\rho}-\vec{r}_4\right) d^3r d^3\rho d^3r_3 d^3r_4$$

$$= \int \left[\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)} \delta\left(\frac{1}{2}\vec{r}+\vec{\rho}-\vec{r}_4\right) d^3r_4\right] d^3r d^3\rho d^3r_3$$

$$= \int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})^2+\left(\frac{1}{2}\vec{r}+\vec{\rho}-\vec{\rho}\right)^2\right]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).\left(\frac{1}{2}\vec{r}+\vec{\rho}-\vec{r}_3\right)} d^3r d^3\rho d^3r_3$$

$$= \int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})^2+\left(\frac{\vec{r}}{2}\right)^2\right]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{\rho}-\vec{r}_3)} e^{i\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right).\vec{r}} d^3r d^3\rho d^3r_3$$

$$= \int e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{\rho}-\vec{r}_3)} \left[\int e^{-\frac{7}{12}\beta^2 r^2} e^{i\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right).\vec{r}} d^3r\right] d^3\rho d^3r_3$$

$$= \int e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{\rho}-\vec{r}_3)} \left[\int e^{-\frac{7\beta^2}{12}\left[r^2-\frac{12i}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right).\vec{r}\right]} d^3r\right] d^3\rho d^3r_3$$

By completing the square for the integral over $\vec{r}$ we get

$$= \int e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{\rho}-\vec{r}_3)} \left[e^{-\frac{3}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right)^2} \int e^{-\frac{7\beta^2}{12}\left[r-\frac{6i}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right)\right]^2} d^3r\right] d^3\rho d^3r_3$$

$$= \int e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{\rho}-\vec{r}_3)} \left[e^{-\frac{3}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right)^2} \left(\frac{1}{\beta}\sqrt{\frac{12\pi}{7}}\right)^3\right] d^3\rho d^3r_3$$

$$= \left(\frac{1}{\beta}\sqrt{\frac{12\pi}{7}}\right)^3 e^{-\frac{3}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right)^2} \int \left[\int e^{-\frac{\beta^2}{3}(\vec{r}_3-\vec{\rho})^2} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{\rho}-\vec{r}_3)} d^3\rho\right] d^3r_3$$

$$= \left(\frac{1}{\beta}\sqrt{\frac{12\pi}{7}}\right)^3 e^{-\frac{3}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right)^2} \int \left[\int e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})^2-\frac{3i}{\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{\rho}-\vec{r}_3)\right]} d^3\rho\right] d^3r_3$$

Again by completing the square for the integral over $\vec{\rho}$ we have

$$= \left(\frac{1}{\beta}\sqrt{\frac{12\pi}{7}}\right)^3 e^{-\frac{3}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right)^2} \int \left[e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2} \int e^{-\frac{\beta^2}{3}\left[(\vec{r}_3-\vec{\rho})-\frac{3i}{2\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)\right]^2} d^3\rho\right] d^3r_3$$

$$= \left(\frac{1}{\beta}\sqrt{\frac{12\pi}{7}}\right)^3 e^{-\frac{3}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right)^2} \int \left[e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2} \left(\frac{1}{\beta}\sqrt{3\pi}\right)^3\right] d^3r_3$$

$$= \left(\frac{1}{\beta}\sqrt{\frac{12\pi}{7}}\right)^3 \left(\frac{1}{\beta}\sqrt{3\pi}\right)^3 e^{-\frac{3}{7\beta^2}\left(\frac{\vec{K}}{6}-\frac{\vec{k}}{2}\right)^2} e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2} \int d^3r_3$$

$$= L^3 \frac{\pi^3}{\beta^6}\left(\frac{6}{\sqrt{7}}\right)^3 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2} \tag{4.24}$$

By substituting eq. (4.24) in eq. (4.23) we obtain

$$\langle v_{14}\rangle = \left(3N^2 - N'^2\right)\frac{t_0}{2L^3}\left(1+\frac{x_0}{2}\right)\left\{\frac{1}{\sqrt{27}}\left(\frac{6}{\sqrt{7}}\right)^3 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\right\} \tag{4.25}$$

We will obtain the same result after evaluating the second term in eq. (4.21), thus

$$\langle v_{24}\rangle = \langle v_{14}\rangle \tag{4.26}$$

Now let us evaluate $\langle \Psi_{total}|v_{34}|\Psi_{total}\rangle$ using the same above technique

$\langle v_{34}\rangle = \langle \Psi_{total}|v_{34}|\Psi_{total}\rangle$

$$= \frac{3}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti*}(34)\psi_{spin}^{t}(34)|\ v_{34}|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti}(34)\psi_{spin}^{t}(34)\rangle$$

$$+\frac{1}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym*}(34)\psi_{spin}^{s}(34)|\ v_{34}|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym}(34)\psi_{spin}^{s}(34)\rangle$$

$$= \frac{3}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti*}(34)\psi_{spin}^{t}\left|-t_0\big(1+x_0P_\sigma(34)\big)\delta(\vec{r}_3-\vec{r}_4)\right|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{anti}(34)\psi_{spin}^{t}(34)\rangle$$

$$+\frac{1}{4}\langle\psi_{space}^{sym*}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym*}(34)\psi_{spin}^{s}(34)\left|-t_0\big(1+x_0P_\sigma(34)\big)\delta(\vec{r}_3-\vec{r}_4)\right|$$

$$\psi_{space}^{sym}(12)\psi_{spin}^{s}(12)\psi_{space}^{sym}(34)\psi_{spin}^{s}(34)\rangle$$

Here we can see that the two neutrons (particle 3 and particle 4) are100% in the triplet state in the first part and 100% in the singlet state in the second part, and so the operator $P_\sigma(34)$, which exchanges the spins of the two neutrons, will give $+1$ in the first part and -1 in the second part. Thus, the term $\langle v_{34}\rangle$ will be

$$= -\frac{3}{4}t_0(1+x_0)\int \psi_{space}^{sym*}(12)\psi_{space}^{anti*}(34)\delta(\vec{r}_3-\vec{r}_4)$$

$$\psi_{space}^{sym}(12)\psi_{space}^{anti}(34)d^3r d^3\rho d^3r_3 d^3r_4$$

$$-\frac{1}{4}t_0(1-x_0)\int \psi_{space}^{sym*}(12)\psi_{space}^{sym*}(34)\ \delta(\vec{r}_3-\vec{r}_4)$$

$$\psi_{space}^{sym}(12)\psi_{space}^{sym}(34)d^3r d^3\rho d^3r_3 d^3r_4$$

Similar to the above technique this equals to

$$= \left(3N^2(1+x_0) - N'^2(1-x_0)\right)\frac{t_0}{2}\frac{1}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \times$$

$$\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]}\, e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}\delta(\vec{r}_3-\ \vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4$$

This integral can be evaluated as in the case of $\langle v_{14}\rangle$ and so we have

$$\langle v_{34}\rangle = \left(3N^2(1+x_0) - N'^2(1-x_0)\right)\frac{t_0}{2L^3} \qquad (4.27)$$

Now, we can rewrite eq. (4.21) in terms of eqs. (4.26) and (4.27) to be

$$\langle \Psi_{total}|\ v_{14}+v_{24}+v_{34}|\Psi_{total}\rangle = \left(3N^2 - N'^2\right)\frac{t_0}{L^3}\left(1+\frac{x_0}{2}\right)\left\{\frac{1}{\sqrt{27}}\left(\frac{6}{\sqrt{7}}\right)^3 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\right\}$$

$$+\left(3N^2(1+x_0) - N'^2(1-x_0)\right)\frac{t_0}{2L^3} \qquad (4.28)$$

Now, we want to find the terms which represent the three-body interactions

$$\langle \Psi_{total}|v_{124}+v_{134}+v_{234}|\Psi_{total}\rangle = \langle \Psi_{total}|v_{124}|\Psi_{total}\rangle + \langle \Psi_{total}|v_{134}|\Psi_{total}\rangle$$

$$+\langle \Psi_{total}|v_{234}|\Psi_{total}\rangle \qquad (4.29)$$

The first term is

$$\langle v_{124}\rangle = \langle \Psi_{total}|v_{124}|\Psi_{total}\rangle$$

$$= \frac{3}{4}\langle \psi^{sym*}_{space}(12)\psi^{s}_{spin}(12)\psi^{anti*}_{space}(34)\psi^{t}_{spin}(34)|\ v_{124}|$$

$$\psi^{sym}_{space}(12)\psi^{s}_{spin}(12)\psi^{anti}_{space}(34)\psi^{t}_{spin}(34)\rangle$$

$$+\frac{1}{4}\langle \psi^{sym*}_{space}(12)\psi^{s}_{spin}(12)\psi^{sym*}_{space}(34)\psi^{s}_{spin}(34)|\ v_{124}|$$

$$\psi^{sym}_{space}(12)\psi^{s}_{spin}(12)\psi^{sym}_{space}(34)\psi^{s}_{spin}(34)\rangle$$

Using the same technique in deriving eq.(4.23) we get

$$\langle v_{124}\rangle = \left(N'^2 - 3N^2\right)\frac{1}{2}\frac{1}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \times$$

$$\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]}\ e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}(\ v_{124})d^3r d^3\rho d^3r_3 d^3r_4$$

Using the three-body Skyrme interaction as in eq. (2.14) we obtained

$$\langle v_{124}\rangle = \left(N'^2 - 3N^2\right)\frac{1}{2}\frac{1}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \times$$

$$\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}\ t_3\delta(\vec{r}_1-\vec{r}_2)\delta(\vec{r}_2-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4$$

$$\langle v_{124}\rangle = \left(N'^2 - 3N^2\right)\frac{t_3}{2}\frac{1}{L^3}\frac{\beta^3}{\pi^{3/2}} e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2} \qquad (4.30)$$

Using the same procedure we can evaluate $\langle v_{134}\rangle$ and $\langle v_{234}\rangle$, and thus we have

$$\langle v_{234}\rangle = \langle v_{134}\rangle = \left(N'^2 - 3N^2\right)\frac{1}{2}\frac{1}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \times$$

$$\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]}\ e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}(\ v_{134})d^3r d^3\rho d^3r_3 d^3r_4$$

$$= \left(N'^2 - 3N^2\right)\frac{1}{2}\frac{1}{\sqrt{27}L^6}\left(\frac{\beta}{\sqrt{\pi}}\right)^6 \times$$

$$\int e^{-\frac{\beta^2 r^2}{2}} e^{-\frac{\beta^2}{3}[(\vec{r}_3-\vec{\rho})^2+(\vec{r}_4-\vec{\rho})^2]} e^{i\left(\frac{\vec{K}}{3}-\vec{k}\right).(\vec{r}_4-\vec{r}_3)}\ t_3\delta(\vec{r}_1-\vec{r}_3)\delta(\vec{r}_3-\vec{r}_4)d^3r d^3\rho d^3r_3 d^3r_4$$

Thus,

$$\langle v_{234}\rangle = \langle v_{134}\rangle = \left(N'^2 - 3N^2\right)\frac{t_3}{2\sqrt{8}}\frac{1}{L^3}\frac{\beta^3}{\pi^{3/2}} \qquad (4.31)$$

By substituting eqs. (4.30) and (4.31) in eq. (4.29) we get

$$\langle \Psi_{total} | v_{124} + v_{134} + v_{234} | \Psi_{total} \rangle = \left(N'^2 - 3N^2 \right) \frac{t_3}{2} \frac{1}{L^3} \frac{\beta^3}{\pi^{3/2}} e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}$$

$$+ \left(N'^2 - 3N^2 \right) \frac{t_3}{\sqrt{8}} \frac{1}{L^3} \frac{\beta^3}{\pi^{3/2}} \qquad (4.32)$$

The total energy of the ^{3}He-neutron system can be written now by substituting eqs. (4.28) and (4.32) in eq. (4.20) we get

$$E_{total} = -B_0 + \frac{\hbar^2 K^2}{2M} + \frac{\hbar^2 k^2}{2m} + \left(3N^2 - N'^2\right) \frac{t_0}{L^3} \left(1 + \frac{x_0}{2}\right) \left\{ \frac{1}{\sqrt{27}} \left(\frac{6}{\sqrt{7}} \right)^3 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2} \right\}$$

$$+ \left(3N^2 (1 + x_0) - N'^2 (1 - x_0) \right) \frac{t_0}{2L^3}$$

$$+ \left(N'^2 - 3N^2 \right) \frac{t_3}{2} \frac{1}{L^3} \frac{\beta^3}{\pi^{3/2}} e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}$$

$$+ \left(N'^2 - 3N^2 \right) \frac{t_3}{\sqrt{8}} \frac{1}{L^3} \frac{\beta^3}{\pi^{3/2}}$$

As mentioned above the terms $\frac{\hbar^2 K^2}{2M}$ and $\frac{\hbar^2 k^2}{2m}$ represents the kinetic energy of the ^{3}He nucleus and the free neutron respectively, and so these terms do not contribute to the binding energy of the system. Thus the binding energy of ^{3}He nucleus plus a free neutron can be written as

$$B = B_0 - \left(3N^2 - N'^2\right) \frac{t_0}{L^3} \left(1 + \frac{x_0}{2}\right) \left\{ \frac{1}{\sqrt{27}} \left(\frac{6}{\sqrt{7}} \right)^3 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2} \right\}$$

$$- \left(3N^2 (1 + x_0) - N'^2 (1 - x_0) \right) \frac{t_0}{2L^3}$$

$$-\left(N'^2 - 3N^2\right)\frac{t_3}{2}\frac{1}{L^3}\frac{\beta^3}{\pi^{3/2}}e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}$$

$$-\left(N'^2 - 3N^2\right)\frac{t_3}{\sqrt{8}}\frac{1}{L^3}\frac{\beta^3}{\pi^{3/2}} \tag{4.33}$$

In the derivation of eq. (4.33) we assumed that there is only one neutron around the ^{3}He nucleus. We just use this assumption for simplicity, but our problem is to show how the presence of a vapor of protons and neutrons will affect the binding energy of the ^{3}He cluster. For very low vapor densities, the free nucleons can be treated as an ideal gas and they do not interact with each other and so their effect on the binding energy of the nucleus can be obtained by multiplying the above terms by the number of nucleons. If we have n protons and neutrons (not only one neutron) around the ^{3}He nucleus, the density of this vapor of protons and neutrons will be $\rho = \frac{n}{L^3}$. Now we can write the binding energy of ^{3}He nucleus surrounded by a vapor of protons and neutrons as

$$B = B_0 - n\left(3N^2 - N'^2\right)\frac{t_0}{L^3}\left(1+\frac{x_0}{2}\right)\left\{\frac{1}{\sqrt{27}}\left(\frac{6}{\sqrt{7}}\right)^3 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\right\}$$

$$-n\left(3N^2(1+x_0) - N'^2(1-x_0)\right)\frac{t_0}{2L^3}$$

$$-n\left(N'^2 - 3N^2\right)\frac{t_3}{2}\frac{1}{L^3}\frac{\beta^3}{\pi^{3/2}}e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}$$

$$-n\left(N'^2 - 3N^2\right)\frac{t_3}{\sqrt{8}}\frac{1}{L^3}\frac{\beta^3}{\pi^{3/2}}$$

But as mentioned above $\rho = \frac{n}{L^3}$ so

$$B = B_0 - \rho\left(3N^2 - N'^2\right) t_0 \left(1 + \frac{x_0}{2}\right)\left\{\frac{1}{\sqrt{27}}\left(\frac{6}{\sqrt{7}}\right)^3 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}\right\}$$

$$-\rho\left(3N^2(1 + x_0) - N'^2(1 - x_0)\right)\frac{t_0}{2}$$

$$-\rho\left(N'^2 - 3N^2\right)\frac{t_3}{2}\frac{\beta^3}{\pi^{3/2}} e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}$$

$$-\rho\left(N'^2 - 3N^2\right)\frac{t_3}{\sqrt{8}}\frac{\beta^3}{\pi^{3/2}} \tag{4.34}$$

By substituting N^2 and N'^2 from eq. (4.13) in eq. (4.34) we get

$$B = B_0 - \rho t_0 \left(1 + \frac{x_0}{2}\right)\left\{\frac{1}{\sqrt{27}}\left(\frac{6}{\sqrt{7}}\right)^3 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}\right\}$$

$$-\rho\left(\frac{3}{2}(1 + x_0) - \frac{1}{2}(1 - x_0)\right)\frac{t_0}{2} + \rho\frac{t_3}{2}\frac{\beta^3}{\pi^{3/2}} e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}$$

$$+\rho\frac{t_3}{\sqrt{8}}\frac{\beta^3}{\pi^{3/2}}$$

Finally this formula can be written simply as:

$$B = B_0 + \rho\left\{-\left(\frac{1}{2} + x_0\right) t_0 - \left(1 + \frac{x_0}{2}\right)\frac{1}{\sqrt{27}}\left(\frac{6}{\sqrt{7}}\right)^3 t_0 e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2} + \frac{1}{\sqrt{8}}\frac{\beta^3}{\pi^{3/2}} t_3 + \right.$$

$$\left.\frac{1}{2}\frac{\beta^3}{\pi^{3/2}} t_3 e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3} - \vec{k}\right)^2}\right\} \tag{4.35}$$

We must now take the ensemble average of this quantity over all values of $\vec{K}$ and $\vec{k}$ in order to get the expectation value of B. We will do this in the next chapter by using the Fermi-Dirac statistics for the nucleons and the ^{3}He nuclei. We will assume thermal and chemical equilibrium and use the Nuclear Statistical Equilibrium (NSE) model to relate the chemical potential of the ^{3}He nuclei to the chemical potential of the nucleons.

CHAPTER 5. IDEAL FERMI GAS STATISTICS

As motioned in chapter 2 both protons and neutrons are fermions. A ^{3}He nucleus is also a fermion as its spin is $1/2$. Thus, we will use Fermi-Dirac statistics to describe the momentum distribution of protons, neutrons and ^{3}He nuclei within the box. We will assume that we have an ideal Fermi gas of nucleons in thermal and chemical equilibrium with the ^{3}He nuclei.

5.1 IDEAL FERMI GAS

An ideal Fermi gas consists of non-interacting indistinguishable fermions which in turn must obey the Pauli exclusion principle. For a Fermi system the average occupancy number of a single-particle level with energy ε is give by the Fermi-Dirac distribution function [45]

$$f_{FD}(\varepsilon) = \frac{1}{\exp((\varepsilon - \mu)/k_b T) + 1} \tag{5.1}$$

where μ is the chemical potential which is a function of density ρ and temperature T. The Boltzmann constant $k_b = 1.380 \times 10^{-23} J/K = 8.617 \times 10^{-11}$ (MeV$/K$). From eq. (5.1) we can notice that the average occupancy number cannot be more than 1 or less than 0. We can also notice that the Fermi-Dirac distribution has a special behavior at absolute zero $(T = 0)$. At $T = 0$ the occupation is unity for all states with $\varepsilon < \mu$ and is zero for all states with $\varepsilon > \mu$. This means that the Fermi gas at absolute zero can be described as a completely degenerate gas. The value of μ at $T = 0$ is often called the Femi-energy ε_f. Thus, at absolute zero eq. (5.1) will be

$$f(\varepsilon) = \begin{cases} 1 & for \ \varepsilon < \varepsilon_f \\ 0 & for \ \varepsilon > \varepsilon_f \end{cases} \tag{5.2}$$

If we consider an ideal Fermi system of n non-interacting particles in a cubical box of volume $V = L^3$, such as a system of non-interacting nucleons which we will use in our work, the total number of particles n of such a system is

$$n = g \sum_i \frac{1}{\exp((\varepsilon_i - \mu)/k_b T) + 1} \tag{5.3}$$

where g is a weight factor arising from the internal structure of the particles (such as the spin). For nucleons this factor represents the spin-isospin degeneracy factor and so $g = 4$. For such a system the single particle energy is given by

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} \tag{5.4}$$

where m is the mass of the particle, and k is the wave number of the particle.

Talahmeh and Jaqaman in their work [14] started from eq. (5.3) and derived the general form of the equation of state at temperature T and number density $\rho = \frac{n}{V}$, of an infinite system of non-interacting particles, which is

$$\mu(T, \rho) = k_b T \left(ln \left(\frac{\lambda_T^3 \rho}{g} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{\lambda_T^3 \rho}{g} \right)^n \right) \tag{5.5}$$

where $\lambda_T^3 = \left(\frac{2\pi\hbar^2}{mk_bT}\right)^{3/2}$ is called the thermal wavelength of the gas particles. Also in eq. (5.5) the $b_n's$ are the expansion coefficients that were obtained by using the method of series inversion. From this equation one can notice easily the dependence of the chemical potential upon density and temperature.

In our work we will use the first six b coefficients which were evaluated in [14] using MATLAB. These coefficients are listed in Table 4 below

Table 4. Numerical values of the b coefficients for an ideal Fermi gas

$\boldsymbol{n}$	$\boldsymbol{b_n}$
$n = 1$	0.3535533905933
$n = 2$	-0.0049500897299
$n = 3$	$1.483857713 \times 10^{-4}$
$n = 4$	$-4.4256301 \times 10^{-6}$
$n = 5$	1.006362×10^{-7}
$n = 6$	-4.272×10^{-10}

Eq. (5.5) represents the general form of the equation of state of any ideal Fermi gas. In our work we have an ideal Fermi gas of nucleons and an ideal Fermi gas of ^{3}He nuclei. As mentioned in the previous chapter we will assume thermal and chemical equilibrium and use the NSE model to relate the chemical potential of the ^{3}He nuclei to the chemical potential of the nucleons. We will do this in the next section.

5.2 NUCLEAR STATISTICAL EQUILIBRIUM (NSE) MODEL

The Nuclear Statistical Equilibrium (NSE) model is one of the models used to derive the equation of state of nuclear matter at the low density limit from a statistical point of view. It ignores the in-medium effects which result in the dissolution of clusters into their components. This means that this model fails at high densities where the medium modifications are important [10, 12].

As we mentioned above, the main problem in our research is to study the stability of ^{3}He nuclei immersed in a hot low-density vapor of symmetric nuclear matter of protons and neutrons. We can see now that the NSE model is convenient to be used in our work.

The chemical potentials of the surrounding protons and neutron (μ_p and μ_n respectively) are given by eq. (5.5). According to the NSE model there will be a statistical and chemical equilibrium between clusters and the surrounding vapor of protons and neutrons, and so the chemical potential of a cluster C can be written as follows

$$\mu_C = Z\mu_p + N\mu_n \tag{5.6}$$

But we assumed that the surrounding nuclear matter is symmetric. As a result of this symmetry assumption $\mu_p = \mu_n = \mu_k$, and thus

$$\mu_C = A\mu_k \tag{5.7}$$

where $A = Z + N$ is the atomic number of the cluster. In view of eq. (5.7) it is easy now to write the chemical potential of ^{3}He cluster in terms of the chemical potential of the vapor of nucleons

$$\mu_{^3He} = 3\mu_k \tag{5.8}$$

Now we must pay attention to the probability of finding a cluster C with energy ε_C as it is slightly different from that of free nucleons. We will use the Fermi-Dirac distribution function defined in eq. (5.1) but for the case of a cluster containing A nucleons bound to each other, the binding energy of the cluster B_C must be subtracted from the energy ε_C. This is clear in eq. (5.9)

$$f_{FD}(\varepsilon_C) = \frac{1}{\exp((\varepsilon_C - B_C - \mu_C)/k_b T) + 1} \tag{5.9}$$

where B_C is the density-dependent binding energy as given by eq. (4.35).

By making use of the above discussion in this section and the previous one we can now find the ensemble average over all values of $\vec{K}$ and $\vec{k}$ and so the expectation value of B as we will see in the next section.

5.3 Ensemble Average Over All Values of $\vec{K}$ And $\vec{k}$ At $T > 0$

It is known from statistical mechanics that the thermal average for any quantity $X(p)$ is given by the relation

$$\langle X(p)\rangle = \frac{\iint X(p)f(p)d^3p}{\iint f(p)d^3p} \tag{5.10}$$

where $f(p)$ is probability function and p represent the momentum.

In our work the probability function is the Fermi-Dirac distribution function (see eq. 5.1). Let us now use eq. (5.10) to find $\langle e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle$ and $\langle e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle$. We will find the average for the general case of $\langle e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle$ where α is constant, then

$$\langle e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle = \frac{\left[2\left(\frac{L}{2\pi}\right)^3\right]\left[4\left(\frac{L}{2\pi}\right)^3\right]\iint f_{FD}(K)f_{FD}(k)e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}d^3Kd^3k}{\left[2\left(\frac{L}{2\pi}\right)^3\right]\left[4\left(\frac{L}{2\pi}\right)^3\right]\iint f_{FD}(K)f_{FD}(k)d^3Kd^3k}$$

where the factor 2 and 4 are the degeneracy factors: the first one stands for the two spin states of ^{3}He nucleus and the second one means there are two states of spin and isospin of nucleon. The average will be

$$\langle e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle = \frac{\iint f_{FD}(K)f_{FD}(k)e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}d^3Kd^3k}{\iint f_{FD}(K)f_{FD}(k)d^3Kd^3k} \tag{5.11}$$

Let us find the integral in the numerator at first

$$\iint f_{FD}(K)f_{FD}(k)e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}d^3Kd^3k$$

$$= \iint f_{FD}(K)f_{FD}(k)e^{-\alpha\left(\frac{K^2}{9}+k^2-\frac{2}{3}\vec{K}.\vec{k}\right)}d^3Kd^3k$$

$$= \int f_{FD}(K)\, e^{-\alpha\frac{K^2}{9}} \left[\int f_{FD}(k) e^{-\alpha\left(k^2 - \frac{2}{3}\vec{K}.\vec{k}\right)} d^3k\right] d^3K \qquad (5.12)$$

Let

$$J = \int f_{FD}(k) e^{-\alpha\left(k^2 - \frac{2}{3}\vec{K}.\vec{k}\right)} d^3k \qquad (5.13)$$

Let $\vec{K}$ be in the z-direction and so it makes angle θ with $\vec{k}$ (θ is the angle in the spherical coordinates) which means that eq. (5.13) can be written as

$$J = 2\pi \int f_{FD}(k) k^2 e^{-\alpha k^2} \left[\int e^{\frac{2}{3}\alpha K k \cos\theta} \sin\theta \, d\theta\right] dk$$

After evaluating the angular integral we get

$$J = \frac{3\pi}{\alpha K} \int \left[k e^{-\alpha\left(k^2 - \frac{2}{3}Kk\right)} - k e^{-\alpha\left(k^2 + \frac{2}{3}Kk\right)}\right] f_{FD}(k) dk \qquad (5.14)$$

But we know that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \text{where } |x| < 1 \qquad (5.15)$$

So by making use of eq. (5.15) the Fermi-Dirac distribution function can be written as

$$f_{FD}(k) = \frac{1}{\exp((\varepsilon_k - \mu_k)/k_b T) + 1}$$

$$= \frac{\exp(-(\varepsilon_k - \mu_k)/k_b T)}{\exp(-(\varepsilon_k - \mu_k)/k_b T) + 1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \exp(-(\varepsilon_k - \mu_k)/k_b(T/n))$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} f_k(T/n), \tag{5.16}$$

where

$$f_k(T/n) = \exp(-(\varepsilon_k - \mu_k)/k_b(T/n)) \tag{5.17}$$

represents the classical Maxwell-Boltzmann distribution function.

Substituting eq. (5.17) in eq. (5.14) we get

$$= \frac{3\pi}{\alpha K} \int \left[k e^{-\alpha\left(k^2 - \frac{2}{3}Kk\right)} - k e^{-\alpha\left(k^2 + \frac{2}{3}Kk\right)} \right] \times$$

$$\{f_k(T) - f_k(T/2) + f_k(T/3) - f_k(T/4) + f_k(T/5) - \cdots\} dk$$

$$= \left\{ \frac{3\pi}{\alpha K} \int \left[k e^{-\alpha\left(k^2 - \frac{2}{3}Kk\right)} - k e^{-\alpha\left(k^2 + \frac{2}{3}Kk\right)} \right] f_k(T) dk \right.$$

$$- \frac{3\pi}{\alpha K} \int \left[k e^{-\alpha\left(k^2 - \frac{2}{3}Kk\right)} - k e^{-\alpha\left(k^2 + \frac{2}{3}Kk\right)} \right] f_k(T/2) dk$$

$$+ \frac{3\pi}{\alpha K} \int \left[k e^{-\alpha\left(k^2 - \frac{2}{3}Kk\right)} - k e^{-\alpha\left(k^2 + \frac{2}{3}Kk\right)} \right] f_k(T/3) dk$$

$$\left. - \frac{3\pi}{\alpha K} \int \left[k e^{-\alpha\left(k^2 - \frac{2}{3}Kk\right)} - k e^{-\alpha\left(k^2 + \frac{2}{3}Kk\right)} \right] f_k(T/4) dk + \cdots \right\}$$

Now we can write J as

$$J = \frac{3\pi}{\alpha K}\sum_{n=1}^{\infty}(-1)^{n+1}J_n \tag{5.18}$$

where

$$J_n = \int \left[ke^{-\alpha\left(k^2-\frac{2}{3}Kk\right)} - ke^{-\alpha\left(k^2+\frac{2}{3}Kk\right)}\right] f_k(T/n)dk \tag{5.19}$$

This integral can be evaluated as follows

$$J_n = \int \left[ke^{-\alpha\left(k^2-\frac{2}{3}Kk\right)} - ke^{-\alpha\left(k^2+\frac{2}{3}Kk\right)}\right] e^{(-(\varepsilon_k-\mu_k)/k_b(T/n))}\, dk$$

$$= \int \left[ke^{-\alpha\left(k^2-\frac{2}{3}Kk\right)} - ke^{-\alpha\left(k^2+\frac{2}{3}Kk\right)}\right] e^{-n\hbar^2k^2/2mk_bT} e^{n\mu_k/k_bT}\, dk$$

$$= e^{n\mu_k/k_bT}\left\{\int ke^{-\alpha\left(k^2-\frac{2}{3}Kk\right)} e^{-n\hbar^2k^2/2mk_bT}\, dk\right.$$

$$\left. - \int ke^{-\alpha\left(k^2+\frac{2}{3}Kk\right)} e^{-n\hbar^2k^2/2mk_bT}\, dk\right\}$$

$$= e^{n\mu_k/k_bT}\left\{\int ke^{-\left(\alpha+\frac{n\hbar^2}{2mk_bT}\right)k^2+\frac{2}{3}\alpha Kk}\, dk\right.$$

$$\left. - \int ke^{-\left(\alpha+\frac{n\hbar^2}{2mk_bT}\right)k^2+\frac{2}{3}\alpha Kk} dk\right\}$$

$$= e^{n\mu_k/k_bT}\left\{\int ke^{-\left(\frac{2m\alpha k_bT+n\hbar^2}{2mk_bT}\right)\left[k^2-\frac{(4m\alpha k_bTK)}{3(2m\alpha k_bT+n\hbar^2)}k\right]}\, dk\right.$$

$$\left. - \int ke^{-\left(\frac{2m\alpha k_bT+n\hbar^2}{2mk_bT}\right)\left[k^2+\frac{(4m\alpha k_bTK)}{3(2m\alpha k_bT+n\hbar^2)}k\right]}\, dk\right\}$$

Let $a_n = \frac{(2m\alpha k_b T + n\hbar^2)}{2mk_b T}$ and then completing the square with respect to k in both integrals we get

$$J_n = e^{n\mu_k/k_b T} e^{\frac{\alpha^2 K^2}{9a_n}} \left\{ \int ke^{-a_n\left[k - \frac{\alpha K}{3a_n}\right]^2} dk - \int ke^{-a_n\left[k + \frac{\alpha K}{3a_n}\right]^2} dk \right\}$$

Let us use the following change of variables:

$u = k - \frac{\alpha K}{3a_n}$ in the first integral and $u = k + \frac{\alpha K}{3a_n}$ in the second integral, so that

$$J_n = e^{n\mu_k/k_b T} e^{\frac{\alpha^2 K^2}{9a_n}} \left\{ \int_{-\frac{\alpha K}{3a_n}}^{\frac{\alpha K}{3a_n}} u \quad e^{-a_n u^2} du + 2.\frac{\alpha K}{3a_n} \int_0^\infty e^{-a_n u^2} du \right\}$$

Since the first integral between brackets is odd it will vanish and so we get

$$J_n = e^{n\mu_k/k_b T} e^{\frac{\alpha^2 K^2}{9a_n}} \frac{2\alpha K}{3a_n} \frac{1}{2} \sqrt{\frac{\pi}{a_n}}$$

$$= e^{n\mu_k/k_b T} e^{\frac{\alpha^2 K^2}{9a_n}} \frac{\alpha K}{3a_n} \sqrt{\frac{\pi}{a_n}} \tag{5.20}$$

In view of eq. (5.18), eq. (5.12) becomes

$$= \int f_{FD}(K)\, e^{-\frac{\alpha K^2}{9}} \left[\frac{3\pi}{\alpha K} \{J_1 - J_2 + J_3 - J_4 + \cdots\} \right] d^3K$$

$$= 4\pi \int f_{FD}(K)\, e^{-\frac{\alpha K^2}{9}} \left[\frac{3\pi}{\alpha K} \{J_1 - J_2 + J_3 - J_4 + \cdots\} \right] K^2 dK$$

$$= \frac{12\pi^2}{\alpha} \int f_{FD}(K)\, e^{-\frac{\alpha K^2}{9}} \{J_1 - J_2 + J_3 - J_4 + \cdots\} K dK$$

$$= \frac{12\pi^2}{\alpha} \left\{ \int f_{FD}(K)\, e^{-\frac{\alpha K^2}{9}} J_1 K dK - \int f_{FD}(K)\, e^{-\frac{\alpha K^2}{9}} J_2 K dK \right.$$

$$\left. + \int f_{FD}(K)\, e^{-\frac{\alpha K^2}{9}} J_3 K dK - \int f_{FD}(K)\, e^{-\frac{\alpha K^2}{9}} J_4 K dK + \cdots \right\}$$

$$= \frac{12\pi^2}{\alpha} \sum_{n=1} (-1)^{n+1} J_n' \tag{5.21}$$

where

$$J_n' = \int f_{FD}(K)\, e^{-\frac{\alpha K^2}{9}} J_n K dK \tag{5.22}$$

Now in order to evaluate the integral in eq. (5.22) we will use the same expansion for $f_{FD}(K)$ as in eqs. (5.16) and (5.17), thus

$$J_n' = \int f_{FD}(K)\, e^{-\alpha \frac{K^2}{9}} \left[e^{n\mu_k/k_b T} e^{\frac{\alpha^2 K^2}{9a_n}} \frac{\alpha K}{3a_n} \sqrt{\frac{\pi}{a_n}} \right] K dK$$

$$= \frac{\alpha e^{n\mu_k/k_b T}}{3a_n} \sqrt{\frac{\pi}{a_n}} \left\{ \int K^2\, e^{-\left(\frac{\alpha}{9} + \frac{\alpha^2}{9a_n}\right) K^2} \{f_K(T) - f_K(T/2) + \right.$$

$$\left. f_K(T/3) - f_K(T/4) + f_K(T/5) - \cdots\} dK \right\}$$

where $f_K(T/n)$ can be defined as in eq. (5.17)

$$f_K(T/n) = \exp\left(-\left(\varepsilon_K - \mu_{^3He}\right)/k_b(T/n)\right)$$

Since f_K here represents the Fermi-Dirac distribution function of the ^{3}He nuclei we must subtract the binding energy of ^{3}He cluster from ε_K as we did in eq. (5.9). Thus

$$f_K(T/n) = \exp\left(-\left(\varepsilon_K - B_{^3He} - \mu_{^3He}\right)/k_b(T/n)\right)$$

Now, the integral J_n' will be

$$J_n' = \frac{\alpha e^{n\mu_k/k_bT}}{3a_n}\sqrt{\frac{\pi}{a_n}}\sum_{n=1,s=1}(-1)^{n+1}J_{ns}' \tag{5.23}$$

where

$$J_{ns}' = \int K^2\, e^{-\left(\frac{\alpha}{9}-\frac{\alpha^2}{9a_n}\right)K^2} f_K(T/s)dK$$

$$= \left(e^{s\mu_{^3He}/k_bT}\right)\left(e^{sB_{^3He}/k_bT}\right)\int K^2\, e^{-\left(\frac{\alpha}{9}-\frac{\alpha^2}{9a_n}+\frac{s\hbar^2}{6mk_bT}\right)K^2} KdK$$

$$= \left(e^{s\mu_{^3He}/k_bT}\right)\left(e^{sB_{^3He}/k_bT}\right)\frac{\pi^{1/2}}{4}\left(\frac{18mk_bTa_n}{2m\alpha k_bT(a_n-\alpha)+s(3a_n\hbar^2)}\right)^{3/2} \tag{5.24}$$

Now let us evaluate the denominator in eq. (5.11)

$$\iint f_{FD}(K)f_{FD}(k)d^3Kd^3k = \int f_{FD}(K)\, d^3K \int f_{FD}(k)\, d^3k \tag{5.25}$$

Again we will use eqs. (5.16) and (5.17) to evaluate the two integrals $\int f_{FD}(k)\, d^3k$ and $\int f_{FD}(K)\, d^3K$. The first integral gives

$$\int f_{FD}(k)\, d^3k = \int f_k(T)d^3k - \int f_k(T/2)d^3k + \int f_k(T/3)d^3k$$

$$-\int f_k(T/4)d^3k + \cdots$$

$$= \sum_{n=1}(-1)^{n+1} I_n \tag{5.26}$$

where

$$I_n = \int f_k(T/n)d^3k = e^{n\mu_k/k_bT}\int e^{-\frac{n\hbar^2k^2}{2mk_bT}}\,d^3k$$

$$= 4\pi\left(e^{n\mu_k/k_bT}\right)\frac{\pi^{1/2}}{4}\left(\frac{2mk_bT}{n\hbar^2}\right)^{3/2} \tag{5.27}$$

similarly

$$\int f_{FD}(K)\,d^3K = \int f_K(T)d^3K - \int f_K(T/2)d^3K + \int f_K(T/3)d^3K$$

$$- \int f_K(T/4)d^3K + \cdots$$

$$= \sum_{n=1}(-1)^{n+1}\;F_n \tag{5.28}$$

where

$$F_n = \int f_K(T/n)d^3K = \left(e^{n\mu_{3_{He}}/k_bT}\right)\left(e^{nB_{3_{He}}/k_bT}\right)\int e^{-\frac{n\hbar^2K^2}{2Mk_bT}}\,d^3K$$

$$= 4\pi\left(e^{n\mu_{3_{He}}/k_bT}\right)\left(e^{nB_{3_{He}}/k_bT}\right)\frac{\pi^{1/2}}{4}\left(\frac{2Mk_bT}{n\hbar^2}\right)^{3/2} \tag{5.29}$$

But $M = 3m$ so that

$$F_n = 4\pi \left(e^{n\mu_{3_{He}}/k_bT}\right)\left(e^{nB_{3_{He}}/k_bT}\right)\frac{\pi^{1/2}}{4}\left(\frac{6mk_bT}{n\hbar^2}\right)^{3/2} \tag{5.30}$$

Now we can evaluate the average in eq. (5.11) as

$$\langle e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle = \frac{\frac{12\pi^2}{\alpha}\{J_1'-J_2'+J_3'-J_4'\cdots\}}{\{I_1-I_2+I_3-I_4+\cdots\}\{F_1-F_2+F_3-F_4+\cdots\}} \tag{5.31}$$

This is the general form of the average, but in our work we have two values of α (see eq. 4.35):

$$\alpha_1 = \frac{6}{7\beta^2}\ fm^2 \quad \text{and} \quad \alpha_2 = \frac{3}{4\beta^2}\ fm^2 \tag{5.32}$$

In our work and after checking the convergence in the binding energy we will use only the first seven terms from eqs. (5.23) and (5.31) in evaluating the averages $\langle e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle$ and $\langle e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle$ (see Fig 5.2). While checking the convergence of the binding energy we divided the last term of eqs. (5.23) and (5.31) by factor of two in order to reduce the effect of the alternating sign appeared in these equations. The results will be presented in Chapter 6.

5.4 Ensemble Average Over All Values of $\vec{K}$ And $\vec{k}$ At $T = 0$

In this case the Fermi-Dirac distribution function which we will use is that defined in eq. (5.2). We will use the same procedure as in the previous section. Let us start from eq. (5.11). In the case of $T = 0$ we get in the numerator

$$\iint f_{FD}(K) f_{FD}(k) e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2} d^3K d^3k$$

$$= \iint e^{-\alpha\left(\frac{K^2}{9}+k^2-\frac{2}{3}\vec{K}.\vec{k}\right)} d^3K d^3k$$

$$= \int e^{-\frac{\alpha K^2}{9}} \left[\int e^{-\alpha\left(k^2-\frac{2}{3}\vec{K}.\vec{k}\right)} d^3k\right] d^3K \tag{5.33}$$

Let

$$\zeta = \int e^{-\alpha\left(k^2-\frac{2}{3}\vec{K}.\vec{k}\right)} d^3k \tag{5.34}$$

Using the same procedure as in deriving eq. (5.14) from eq. (5.13) we get

$$\zeta = \int_{\emptyset=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{k=0}^{k_f} e^{-\alpha\left(k^2-\frac{2}{3}Kk\cos\theta\right)} k^2 \sin\theta \, dk d\theta d\emptyset$$

$$= \frac{3\pi}{\alpha K}\left[\int_{k=0}^{k_f} k e^{-\alpha\left(k^2-\frac{2}{3}Kk\right)} dk - \int_{k=0}^{k_f} k e^{-\alpha\left(k^2+\frac{2}{3}Kk\right)} dk\right]$$

By completing the square with respect to k in the above two integrals we get

$$= \frac{3\pi}{\alpha K} e^{\frac{\alpha k^2}{9}}\left[\int_{k=0}^{k_f} k e^{-\alpha\left(k-\frac{K}{3}\right)^2} dk - \int_{k=0}^{k_f} k e^{-\alpha\left(k+\frac{K}{3}\right)^2} dk\right]$$

Let us use the following change of variables:

$u = k - \frac{K}{3}$ in the first integral and $u = k + \frac{K}{3}$ in the second integral, so that

$$= \frac{3\pi}{\alpha K} e^{\frac{\alpha k^2}{9}} \left[\int_{-\frac{K}{3}}^{k_f - \frac{K}{3}} \left(u + \frac{K}{3}\right) e^{-\alpha u^2} du - \int_{\frac{K}{3}}^{k_f + \frac{K}{3}} \left(u + \frac{K}{3}\right) e^{-\alpha u^2} du \right]$$

$$= \frac{3\pi}{\alpha K} e^{\frac{\alpha k^2}{9}} \left[\int_{-\frac{K}{3}}^{k_f - \frac{K}{3}} u e^{-\alpha u^2} du - \int_{\frac{K}{3}}^{k_f + \frac{K}{3}} u e^{-\alpha u^2} du \right.$$

$$\left. + \frac{K}{3} \left\{ \int_{-\frac{K}{3}}^{k_f - \frac{K}{3}} e^{-\alpha u^2} du - \int_{\frac{K}{3}}^{k_f + \frac{K}{3}} e^{-\alpha u^2} dk \right\} \right] \qquad (5.35)$$

Recalling that

$$\int x e^{-\alpha x^2} dx = -\frac{1}{2\alpha} e^{-\alpha x^2}$$

Thus eq. (5.35) will be

$$= \frac{3\pi}{2\alpha^2 K} e^{\frac{\alpha k^2}{9}} \left[e^{-\alpha\left(k_f + \frac{K}{3}\right)^2} - e^{-\alpha\left(k_f - \frac{K}{3}\right)^2} \right]$$

$$+ \frac{\pi}{\alpha} e^{\frac{\alpha k^2}{9}} \left\{ \int_{-\frac{K}{3}}^{k_f - \frac{K}{3}} e^{-\alpha u^2} du - \int_{\frac{K}{3}}^{k_f + \frac{K}{3}} e^{-\alpha u^2} dk \right\}$$

But

$$\int_0^x e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \operatorname{erf}\left(x\sqrt{\alpha}\right)$$

where $\mathrm{erf}\left(x\sqrt{\alpha}\right)$ is the error function, see [64]. Thus, the above equation will be

$$= \frac{3\pi}{2\alpha^2 K} e^{\frac{\alpha k^2}{9}} \left[e^{-\alpha\left(k_f + \frac{K}{3}\right)^2} - e^{-\alpha\left(k_f - \frac{K}{3}\right)^2} \right] +$$

$$\frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{3/2} e^{\frac{\alpha k^2}{9}} \left\{ \mathrm{erf}\left(\sqrt{\alpha}\left[k_f - \frac{K}{3}\right]\right) + \mathrm{erf}\left(\sqrt{\alpha}\left[k_f + \frac{K}{3}\right]\right) \right\} \tag{5.36}$$

In view of eq. (5.36), eq. (5.33) will be

$$\frac{6\pi^2}{\alpha^2}\int_0^{K_f} K \left[e^{-\alpha\left(k_f + \frac{K}{3}\right)^2} - e^{-\alpha\left(k_f - \frac{K}{3}\right)^2} \right] dK +$$

$$2\pi \left(\frac{\pi}{\alpha}\right)^{3/2} \int_0^{K_f} K^2 \left[\mathrm{erf}\left(\sqrt{\alpha}\left[k_f - \frac{K}{3}\right]\right) + \mathrm{erf}\left(\sqrt{\alpha}\left[k_f + \frac{K}{3}\right]\right) \right] dK \tag{5.37}$$

Let us use the Taylor expansion of the error function [46]

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \cdots \right) \tag{5.38}$$

We will use only the first six terms from eq. (5.38) in eq. (5.37) as this will give us convergence in the binding energy versus density curve as we will see below. Now eq. (5.37) will be

$$\frac{6\pi^2}{\alpha^2}\int_0^{K_f} K \left[e^{-\alpha\left(k_f + \frac{K}{3}\right)^2} - e^{-\alpha\left(k_f - \frac{K}{3}\right)^2} \right] dK +$$

$$2\pi \left(\frac{\pi}{\alpha}\right)^{3/2} \frac{2}{\sqrt{\pi}} \int_0^{K_f} K^2 \left[\sqrt{\alpha}\left[k_f - \frac{K}{3}\right] - \frac{\left(\sqrt{\alpha}\left[k_f - \frac{K}{3}\right]\right)^3}{3} + \frac{\left(\sqrt{\alpha}\left[k_f - \frac{K}{3}\right]\right)^5}{10} - \frac{\left(\sqrt{\alpha}\left[k_f - \frac{K}{3}\right]\right)^7}{42} \right.$$

$$+\frac{\left(\sqrt{\alpha}\left[k_f-\frac{K}{3}\right]\right)^9}{216}-\frac{\left(\sqrt{\alpha}\left[k_f-\frac{K}{3}\right]\right)^{11}}{1320}+\sqrt{\alpha}\left[k_f+\frac{K}{3}\right]-\frac{\left(\sqrt{\alpha}\left[k_f+\frac{K}{3}\right]\right)^3}{3}+\frac{\left(\sqrt{\alpha}\left[k_f+\frac{K}{3}\right]\right)^5}{10}$$

$$-\frac{\left(\sqrt{\alpha}\left[k_f+\frac{K}{3}\right]\right)^7}{42}+\frac{\left(\sqrt{\alpha}\left[k_f+\frac{K}{3}\right]\right)^9}{216}-\frac{\left(\sqrt{\alpha}\left[k_f+\frac{K}{3}\right]\right)^{11}}{1320}\Bigg]dK$$

$$=\frac{6\pi^2}{\alpha^2}\int_0^{K_f}K\left[e^{-\alpha\left(k_f+\frac{K}{3}\right)^2}-e^{-\alpha\left(k_f-\frac{K}{3}\right)^2}\right]dK+$$

$$\frac{4\pi^2}{\alpha}\int_0^{K_f}K^2\Bigg[2k_f-\frac{\alpha}{3}\left[\left(k_f-\frac{K}{3}\right)^3+\left(k_f+\frac{K}{3}\right)^3\right]+\frac{\alpha^2}{10}\left[\left(k_f-\frac{K}{3}\right)^5+\left(k_f+\frac{K}{3}\right)^5\right]$$

$$-\frac{\alpha^3}{42}\left[\left(k_f-\frac{K}{3}\right)^7+\left(k_f+\frac{K}{3}\right)^7\right]+\frac{\alpha^4}{216}\left[\left(k_f-\frac{K}{3}\right)^9+\left(k_f+\frac{K}{3}\right)^9\right]$$

$$-\frac{\alpha^5}{1320}\left[\left(k_f-\frac{K}{3}\right)^{11}+\left(k_f+\frac{K}{3}\right)^{11}\right]\Bigg]dK \qquad (5.39)$$

The first term in eq. (5.39) can be evaluated as in eq. (5.35) and so we get

$$=-\frac{27\pi^2}{\alpha^3}\left[e^{-\alpha\left(\frac{K_f}{3}+k_f\right)^2}-e^{-\alpha\left(\frac{K_f}{3}-k_f\right)^2}\right]$$

$$-27\left(\frac{\pi}{\alpha}\right)^{5/2}k_f\left[\mathrm{erf}\left(\sqrt{\alpha}\left[\frac{K_f}{3}-k_f\right]\right)+\mathrm{erf}\left(\sqrt{\alpha}\left[\frac{K_f}{3}+k_f\right]\right)\right]$$

$$+\frac{8\pi^2}{\alpha}k_f\frac{{K_f}^3}{3}-\frac{8\pi^2}{3}\left[{k_f}^3\frac{{K_f}^3}{3}+\frac{1}{3}k_f\frac{{K_f}^5}{5}\right]$$

$$+\frac{8\pi^2\alpha}{10}\left[{k_f}^5\frac{{K_f}^3}{3}+\frac{10}{9}{k_f}^3\frac{{K_f}^5}{5}+\frac{5}{81}k_f\frac{{K_f}^7}{7}\right]$$

$$-\frac{8\pi^2\alpha^2}{42}\left[k_f{}^7\frac{K_f{}^3}{3}+\frac{21}{9}k_f{}^5\frac{K_f{}^5}{5}+\frac{35}{81}k_f{}^3\frac{K_f{}^7}{7}+\frac{7}{729}k_f\frac{K_f{}^9}{9}\right]$$

$$+\frac{8\pi^2\alpha^3}{216}\left[k_f{}^9\frac{K_f{}^3}{3}+\frac{36}{9}k_f{}^7\frac{K_f{}^5}{5}+\frac{126}{81}k_f{}^5\frac{K_f{}^7}{7}+\frac{84}{729}k_f{}^3\frac{K_f{}^9}{9}+\frac{9}{6561}k_f\frac{K_f{}^{11}}{11}\right]$$

$$-\frac{8\pi^2\alpha^4}{1320}\left[k_f{}^{11}\frac{K_f{}^3}{3}+\frac{55}{9}k_f{}^9\frac{K_f{}^5}{5}+\frac{330}{81}k_f{}^7\frac{K_f{}^7}{7}+\frac{462}{729}k_f{}^5\frac{K_f{}^9}{9}+\frac{165}{6561}k_f{}^3\frac{K_f{}^{11}}{11}\right.$$

$$\left.+\frac{11}{59049}k_f\frac{K_f{}^{13}}{13}\right] \tag{5.40}$$

From eq. (5.4) we can see that the Fermi momentum can be written as

$$k_f^2=\frac{2m}{\hbar^2}\,\varepsilon_f,$$

In view of eq. (5.8) $\mu_{^3He}=3\mu_k$ and by making use of the fact that $\mu_k=\varepsilon_f$ at $T=0$ and $M=3m$ we get that $K_f=3k_f$, so that eq. (5.40)

$$=-\frac{27\pi^2}{\alpha^3}\left[e^{-4\alpha k_f{}^2}-1\right]-27\left(\frac{\pi}{\alpha}\right)^{5/2}k_f\,\mathrm{erf}\left(2\sqrt{\alpha}k_f\right)+72\frac{\pi^2}{\alpha}k_f{}^4-\frac{336}{5}\pi^2k_f{}^6$$

$$+\frac{2304}{35}\pi^2\alpha k_f{}^8-\frac{77952}{1470}\pi^2\alpha^2k_f{}^{10}+\frac{2967552}{83160}\pi^2\alpha^3k_f{}^{12}-\frac{27168768}{1321320}\pi^2\alpha^4k_f{}^{14} \tag{5.41}$$

But to find the average we want to evaluate eq. (5.11) and so the integral $\iint f_{FD}(K)f_{FD}(k)d^3Kd^3k$ must be evaluated at $T=0$ which gives

$$\iint f_{FD}(K)f_{FD}(k)d^3Kd^3k=\int f_{FD}(K)d^3K\int f_{FD}(k)d^3k$$

$$= (4\pi)^2 \int_0^{K_f} K^2 dK \int_0^{k_f} k^2 dK$$

$$= (4\pi)^2 \frac{K_f^3}{3}\frac{k_f^3}{3}$$

$$= 48\pi^2 {k_f}^6 \tag{5.42}$$

Now the average at $T = 0$ can be obtained by substituting eqs. (5.41) and (5.42) in eq. (5.11), and thus

$$\langle e^{-\alpha\left(\frac{\vec{K}}{3}-\vec{k}\right)^2} \rangle = -\frac{9}{16\alpha^3}\frac{1}{{k_f}^6}\left[e^{-4\alpha {k_f}^2} - 1\right] - \frac{9}{16}\left(\frac{\pi}{\alpha^5}\right)^{1/2}\frac{1}{{k_f}^5}\operatorname{erf}\left(2\sqrt{\alpha}k_f\right)$$

$$+\frac{15}{10}\frac{1}{\alpha}\frac{1}{{k_f}^2} - \frac{14}{10} + \frac{144}{105}\alpha {k_f}^2 - \frac{2436}{2205}\alpha^2 {k_f}^4 + \frac{7728}{10395}\alpha^3 {k_f}^6 - \frac{70752}{165165}\alpha^4 {k_f}^8 \tag{5.43}$$

Again we will use eq.(5.43) to find the averages for both values of α (see eq. 5.32).

In terms of eqs. (5.31) and (5.43) the expectation value of the binding energy in eq. (4.35) will be

$$\langle B \rangle = B_0 + \rho\Bigg\{-\left(\frac{1}{2} + x_0\right)t_0 - \left(1 + \frac{x_0}{2}\right)\frac{1}{\sqrt{27}}\left(\frac{6}{\sqrt{7}}\right)^3 t_0 \langle e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle + \frac{1}{\sqrt{8}}\frac{\beta^3}{\pi^{3/2}}t_3 +$$

$$\frac{1}{2}\frac{\beta^3}{\pi^{3/2}}t_3 \langle e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle \tag{5.44}$$

It is clear from eq. (5.44) that the expectation value of the binding energy of ^{3}He nuclei is a function of the number density ρ of the surrounding vapor. But we can notice from eqs. (5.31) and (5.43) that the average over all values of $\vec{K}$ and $\vec{k}$ depends on the binding energy. This means that in order to calculate the expectation value of the binding energy many iterative operations are performed to achieve self consistency. In our work we stopped the iteration process when the difference between two successive values of the binding energy is $\leq 0.0001\ MeV$.

We used MATLAB to evaluate the eq. (5.44) which shows the expectation value of the binding energy as a function of the number density ρ at different temperatures. This is what we will show in the next chapter.

The convergence in the binding energy at $T = 0$ when we used just the first six terms from eq. (5.38) can be seen in Fig. 5.1 below

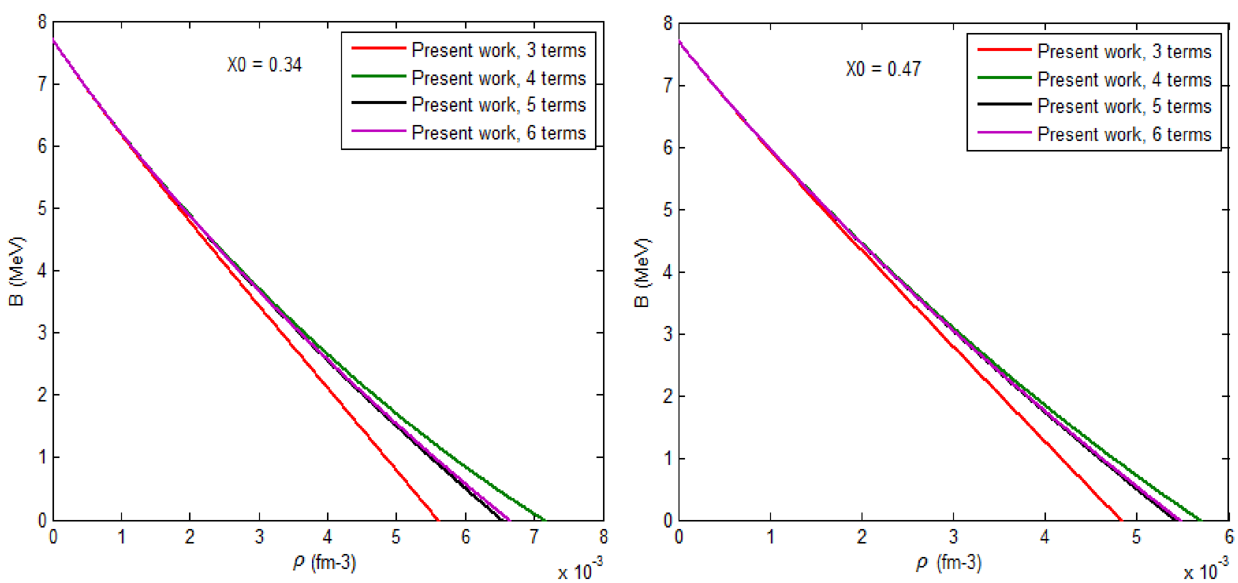
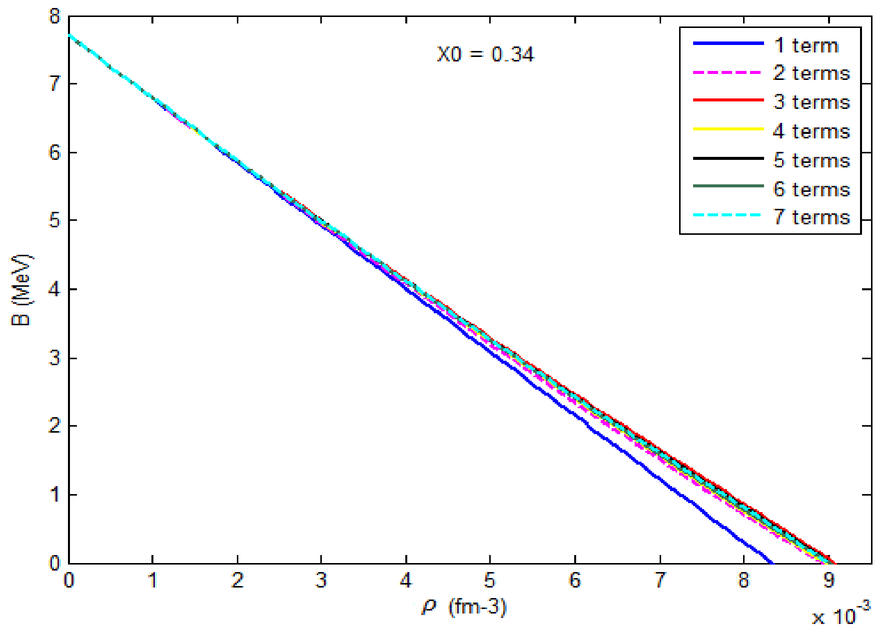
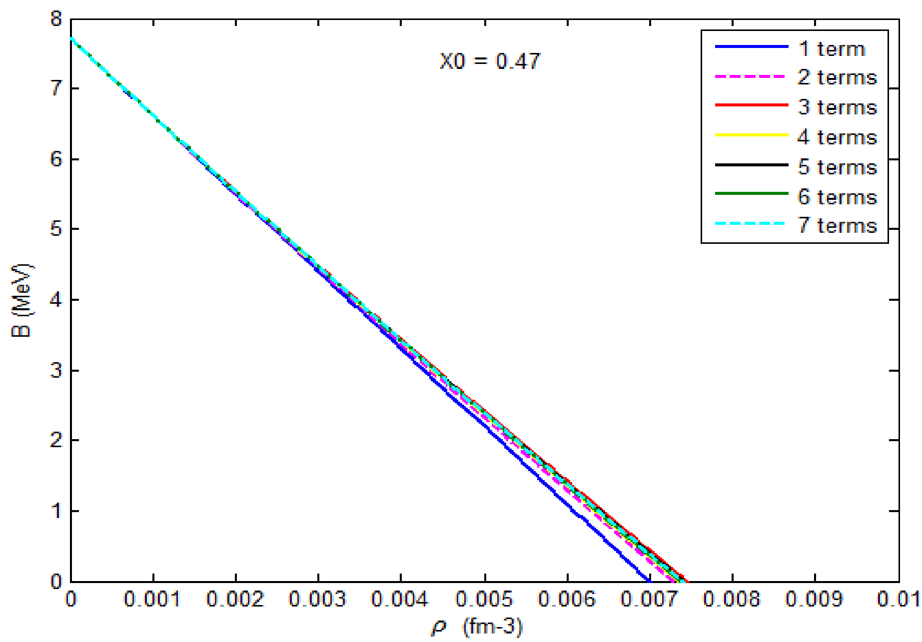


Fig. 5.1. The convergence in the binding energy of ^{3}He nucleus at $T = 0\ MeV$ when we used the first six terms of the Taylor expansion of the Error function



(a)



(b)

Fig. 5.2. The convergence in the binding energy of ^{3}He at $T = 5\ MeV$ when we used the first seven terms from eq. (5.23) and eq. (5.31) for (a) $x_0 = 0.34$ and (b) $x_0 = 0.47$

CHAPTER 6. RESULTS AND CONCLUSIONS

In this chapter we calculate the effect of the presence of nucleons in the surrounding vapor on the binding energy of hot ^{3}He nuclei moving in a hot low-density medium of symmetric nuclear matter. Including the CM momentum of ^{3}He nuclei is the main significant difference between this study and the study proposed by Typel *et. al.* [12].

6.1 BINDING ENERGY DEPENDENCE ON DENSITY AND TEMPERATURE

In the previous chapter we derived the expectation value of the binding energy of a system composed of ^{3}He nuclei moving in a hot low-density symmetric nuclear matter of protons and neutrons (see eq. 5.44). From this equation we can see that the binding energy is a function of the number density ρ of the surrounding medium. The temperature-dependence of the binding energy is included in the terms $\langle e^{-\frac{6}{7\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle$ and $\langle e^{-\frac{3}{4\beta^2}\left(\frac{\vec{K}}{3}-\vec{k}\right)^2}\rangle$ as it is clear in eq. (5.31) and eq. (5.43). We can also see that the expectation value of the binding energy depends on the Skyrme parameters t_0, t_3, and x_0. The values of the first two parameters (t_0 and t_3) were found in chapter three (see eq. 3.36). The value of the parameter x_0 cannot be uniquely determined and so we will allow it to take different values. Here is a list of the values of x_0 which were used in several works in Table 5

Table 5. Different values of x_0

x_0 **value**	**Reference**
0.00	[2]
0.20	[2]
0.34	[16]
0.47	[14]
0.50	[2]
0.56	[16]
0.75	[14]

We will use just two values from the above table in our work. These two values, which are used in [14, 16], are 0.34 and 0.47. The temperature-dependence of the binding energy at the two different values of x_0 is shown in Figs. (6.1) and (6.2) below

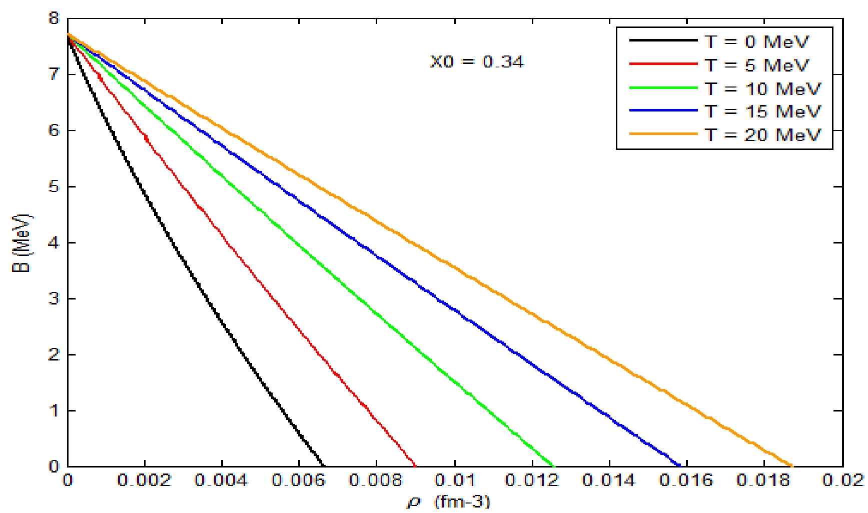


Fig. 6.1. The binding energy of ^3He nucleus in a hot low-density medium of symmetric nuclear matter at different temperatures at $x_0 = 0.34$.

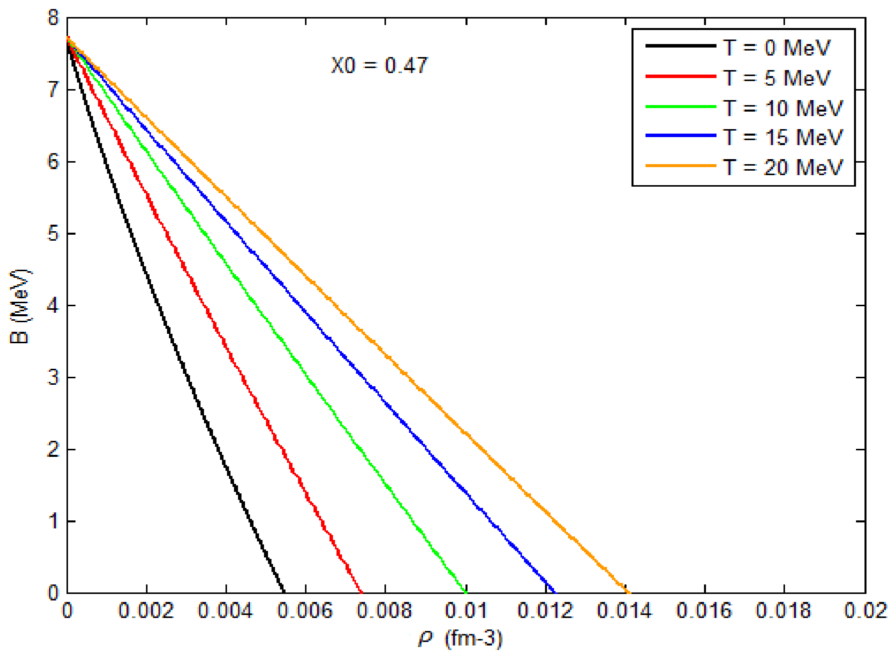


Fig. 6.2. The binding energy of ^{3}He nucleus in a hot low-density medium of symmetric nuclear matter at different temperatures at $x_0 = 0.47$.

From the above two figures we can also notice that regardless the value of x_0, as the temperature increases the cluster can survive up to higher densities. This result is expected as the Pauli blocking effect is less effective at higher temperatures.

Let us now compare our results with those of Typel *et. al*. at each temperature separately. We will show this in the following figures. We will discuss two cases: the case in which the cluster has a momentum $(\vec{K} \neq 0)$ and the case when the cluster is at rest $(\vec{K} = 0)$. In the second case our results are different from those of Typel *et. al.* as we will see below.

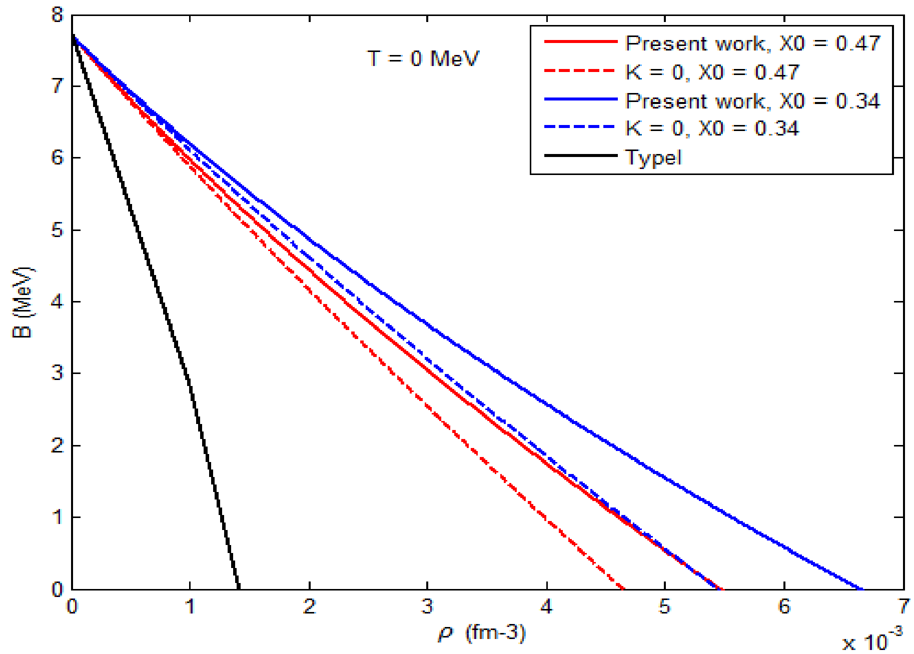


Fig. 6.3. Binding energy versus density at T = 0 MeV

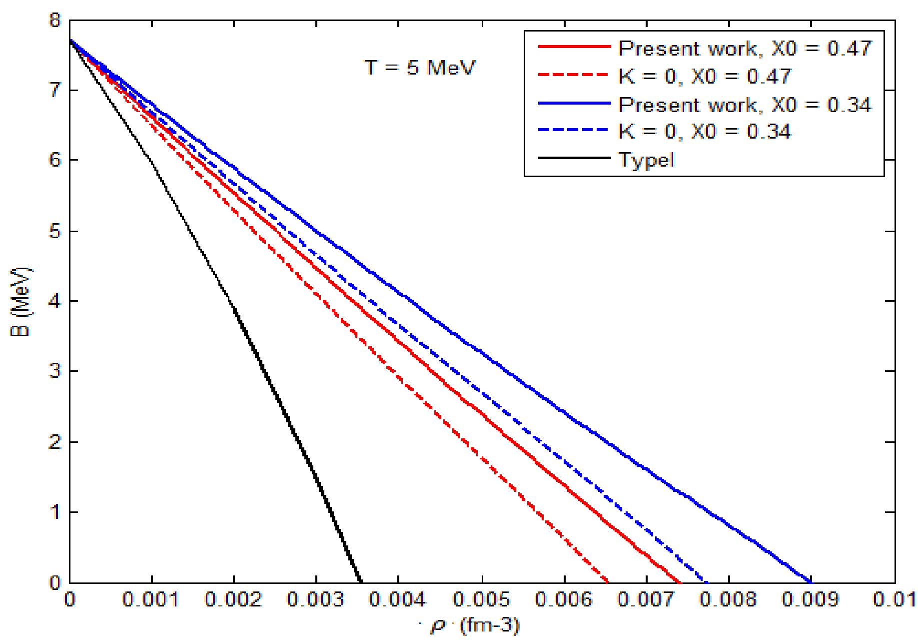


Fig. 6.4. Binding energy versus density at T = 5 MeV

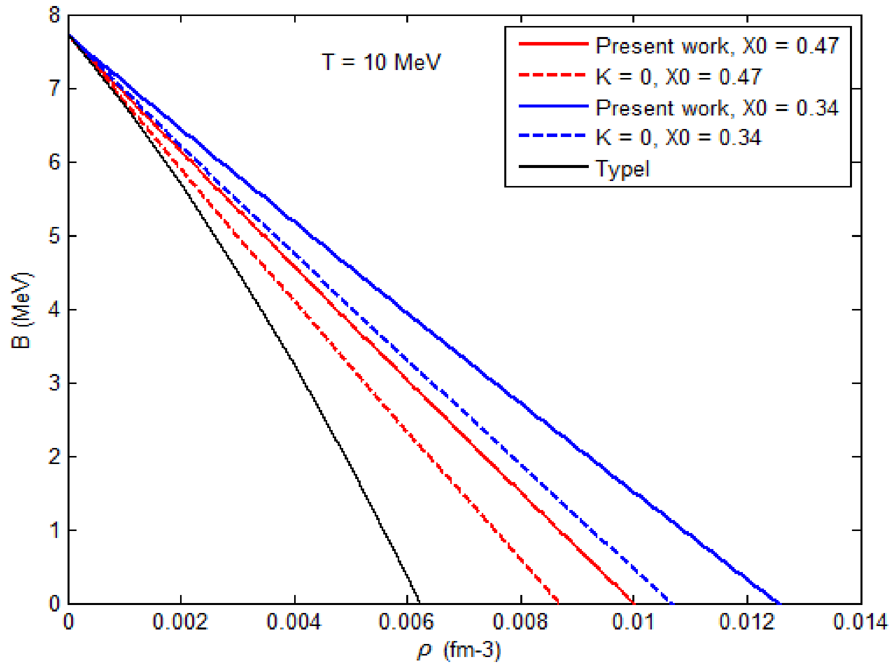


Fig. 6.5. Binding energy versus density at T = 10 MeV

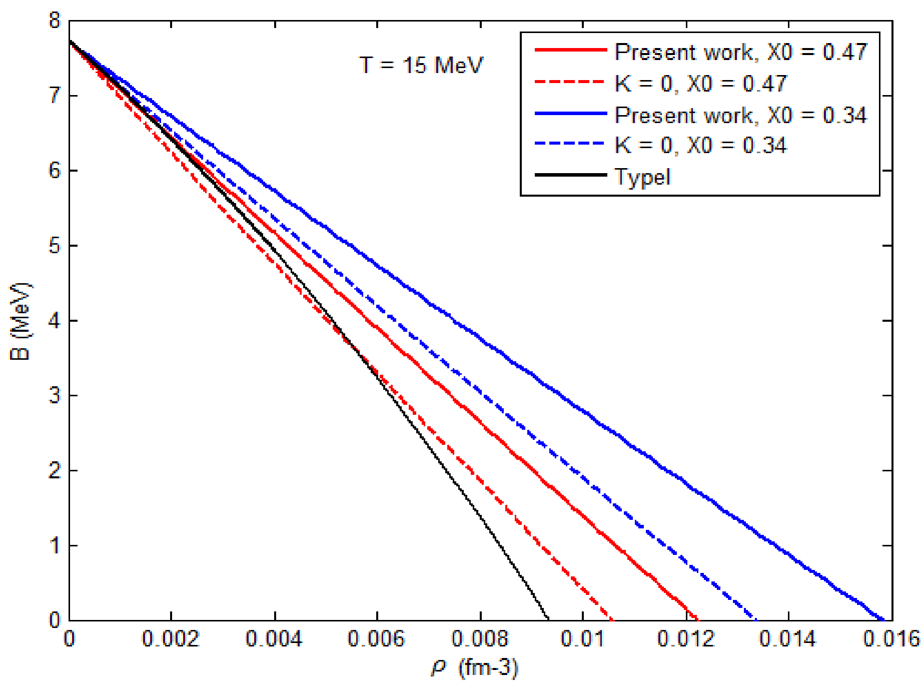


Fig. 6.6. Binding energy versus density at T = 15 MeV

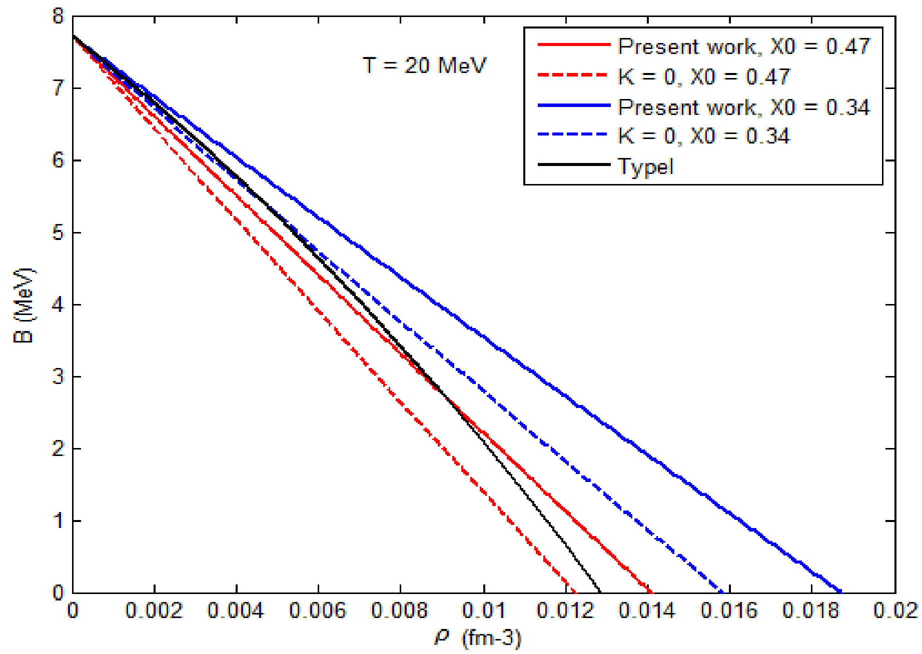


Fig. 6.7. Binding energy versus density at T = 20 MeV

From the above Figs. (6.3 - 6.7) we can conclude the following:

a. Regardless of the temperature, when the ^{3}He nuclei are moving within the surrounding medium (have momentum $\vec{K} \neq 0$) they can survive to higher densities than that if they are at rest $(\vec{K} = 0)$. This is because when $(\vec{K} \neq 0)$ the Pauli blocking effect, which is the main reason for the cluster dissociation, is less effective.

b. Another important conclusion on the above figures is that there is a significant difference between our results when we considered the case in which the ^{3}He clusters are at rest and that of Typel. This because in our work we used some approximations which are different from those in

Typel, for example we assumed that the wave function of the ^{3}He cluster will not change when it is immersed within the medium. We used this assumption to simplify our calculations. Another difference is that we ignored the Coulomb effects which were considered in Typel. We also neglect the self-energy of the cluster. Typel *et. al.* in their work added non-linear terms to the binding energy but we considered just only the linear term. This is clear in Table 7 below.

c. The Skyrme parameter x_0 has also a small effect on the Mott density as we can see. As this parameter cannot be uniquely determined we may have more than one value of the Mott density of ^{3}He nuclei at the same temperature. The values of the Mott density of the ^{3}He nuclei which we obtained in our work are summarized in Table 6

Table 6. Values of the Mott density of ^{3}He nuclei obtained in the present work at different temperatures and different values of x_0

Temperature T (MeV)	$\boldsymbol{x_0 = 0.34}$	$\boldsymbol{x_0 = 0.47}$
	Mott density at $\vec{K} \neq 0$ (nucleon/fm^3)	**Mott density at $\vec{K} \neq 0$ (nucleon/fm^3)**
$\boldsymbol{T = 0}$	0.00663	0.00546
$\boldsymbol{T = 5}$	0.00890	0.00730
$\boldsymbol{T = 10}$	0.01250	0.00990
$\boldsymbol{T = 15}$	0.01580	0.01220
$\boldsymbol{T = 20}$	0.01870	0.01410

Table 7. Comparison between the Mott densities obtained by Typel *et. al*. and those obtained in the present work of the ^{3}He nucleus for different temperatures and with $x_0 = 0.47$

Temperature T (MeV)	$x_0 = 0.47$		
	Mott density (Typel *et. al.* work) (nucleon/fm^3)	**Mott density (Results of present work at $\vec{K} = 0$) (nucleon/fm^3)**	**Mott density (Results of present work at $\vec{K} \neq 0$) (nucleon/fm^3)**
$T = 0$	0.00140	0.00464	0.00546
$T = 5$	0.00360	0.00650	0.00730
$T = 10$	0.00630	0.00870	0.00990
$T = 15$	0.00930	0.01060	0.01220
$T = 20$	0.01280	0.01210	0.01410

In Table 6 we used $x_0 = 0.47$ as the curves we have at this value are closer to those obtained by Typel and his coworkers, and thus, it is more convenient to make the comparison in this case.

Temperature and density dependence of the Mott density of ^{3}He was derived experimentally by Hagel *et. al.* recently [47]. The experimental Mott density of ^{3}He is about 0.0045 nucleon/fm^3 at T = 5 MeV. There is a noticeable difference between this experimental value at the value found in our work. We found that the Mott density of ^{3}He cluster moving with a momentum $\vec{K}$ is about 0.0089 nucleon/fm^3 at $x_0 = 0.34$ and 0.0073 nucleon/fm^3 at $x_0 = 0.47$. When we assumed that ^{3}He is at rest we obtained results closer to the experimental value as we can notice from Table 7. At $x_0 = 0.47$ and $\vec{K} = 0$ the Mott density is 0.0065 nucleon/fm3. The differences between our results and the experimental results are due to the simplifying assumptions we used in our work such as the assumption that the wave function of the ^{3}He cluster will not be affected by the surrounding medium and the neglecting of the self-energy term.

In our work in this thesis we evaluated the change in the binding energy of ^{3}He nuclei due to their CM momentum when immersed in a hot low-density vapor of symmetric nuclear matter of protons and neutrons. We conclude that the contribution from the CM momentum of the clusters is significant and so it must be taken into account in order to be more realistic.

In future work we may develop our work by considering the Coulomb effects and the symmetry energy of the clusters to obtain more realistic results. Another important thing, which we ignored here, is taking into account the change in the wave function of the ^{3}He cluster when it is immersed in a medium. The wave function of the cluster cannot stay the same while its binding energy decreases. We can also improve our work by considering the excited states of ^{3}He especially at high temperatures but this is very complicated and needs numerous work.

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